On the number of forests and trees in large regular graphs

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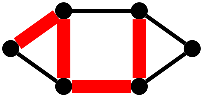
May 12th, 2022.

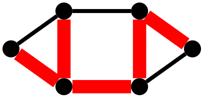
The Plan

- Core definitions and their relation to the Tutte-polynomial
- Motivation
- Extremality results and related conjectures
- Some words about the proof

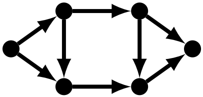
Tutte polynomial



(a) Forest,
number of them $F(G)$



(b) Spanning tree,
number of them $\tau(G)$



(c) Acyclic orientation,
number of them: $a(G)$

$$T_G(x, y) = \sum_{A \subseteq E} (x-1)^{k(A)-k(E)} (y-1)^{k(A)+|A|-v(G)},$$

where $k(A)$ denotes the number of connected components of the graph (V, A), and $v(G)$ denotes the number of vertices of the graph G.

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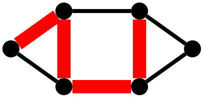

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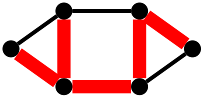

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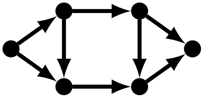

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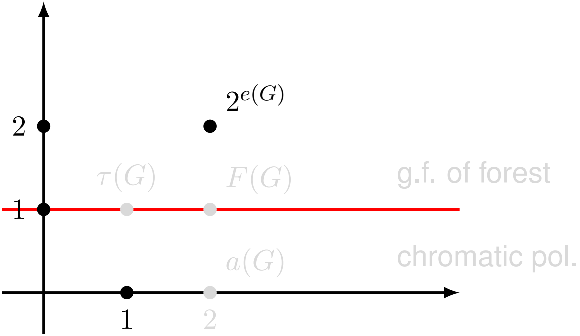

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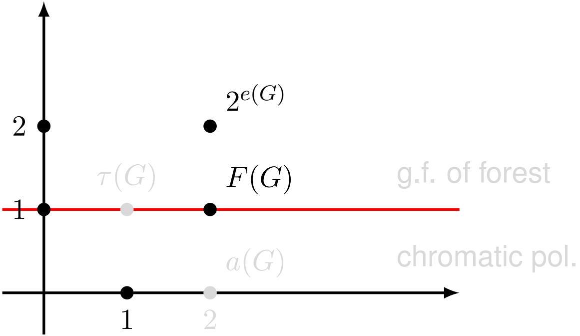

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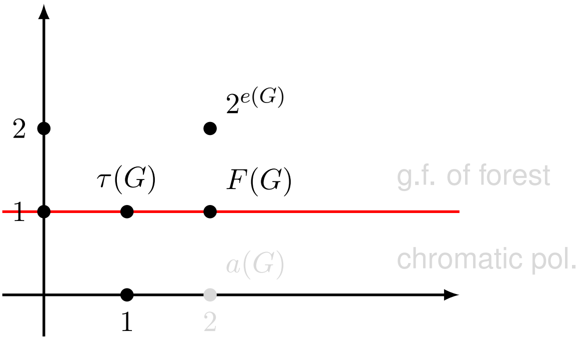

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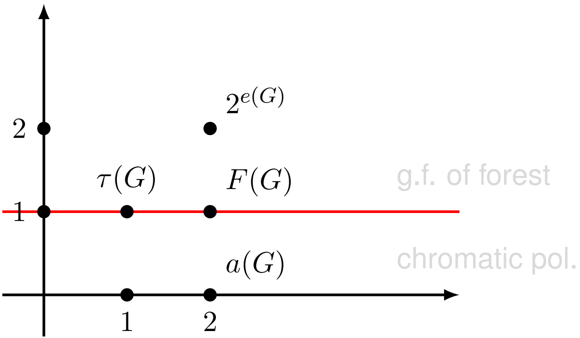

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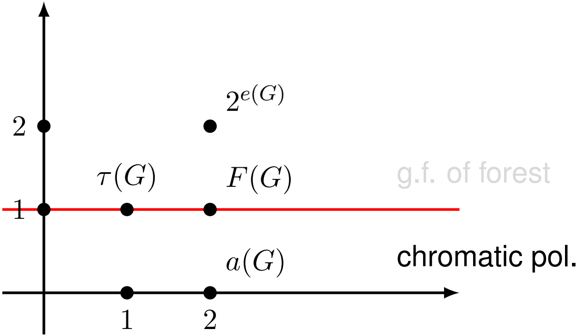

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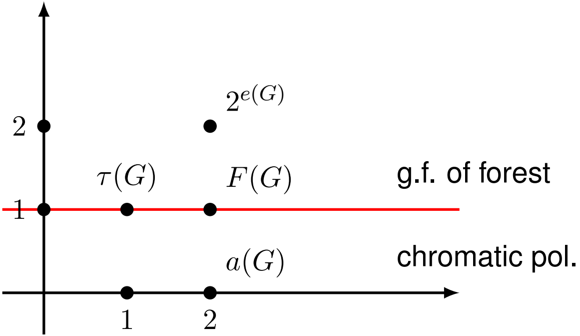

Spanning trees of large girth graphs

Theorem (McKay, Lyons)

For a graph H let $g(H)$ denote the length of the shortest cycle, it is called the girth of the graph H.
Let $(G_n)_n$ be a sequence of connected d-regular graphs such that $\lim_{n\to\infty} g(G_n) = \infty$. Then

$$\lim_{n\to\infty} \tau(G_n)^{1/v(G_n)} = \frac{(d-1)^{d-1}}{(d^2-2d)^{d/2-1}}.$$

What is special about trees?

Features:

- **Matrix-Tree Theorem:** If G is a regular graph, then $\tau(G)$ can be expressed via the eigenvalues of G
- The behavior of the eigenvalues of G_n (actually their distribution) is well understood.

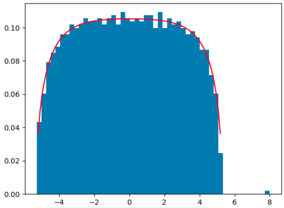

Figure: Eigenvalues of a randomly chosen 8-regular graph on $n = 5000$ vertices.

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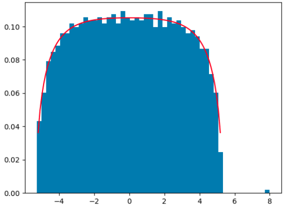

Figure: Eigenvalues of a randomly chosen 8-regular graph on $n = 5000$ vertices.

Limit for evaluations of the Tutte polynomial

Theorem (Bencs and Csikvári)

Let $x \geq 1$ *and* $0 \leq y \leq 1$. *Let* $d \geq 2$, *and let* $(G_n)_n$ *be a sequence of* d*-regular graphs such that* $\lim_{n\to\infty} g(G_n) = \infty$. *Then*

$$\lim_{n\to\infty} T_{G_n}(x,y)^{1/v(G_n)} = t_d(x,y),$$

where

$$t_d(x,y) = \begin{cases} (d-1)\left(\frac{(d-1)^2}{(d-1)^2-x}\right)^{d/2-1} & \text{if } x \leq d-1, \\ x\left(1+\frac{1}{x-1}\right)^{d/2-1} & \text{if } x > d-1. \end{cases}$$

If $(G_n)_n$ *is a sequence of random* d*-regular graphs, then the same statement holds true asymptotically almost surely.*

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Forests and acyclic orientations

Theorem (Bencs and Csikvári)

Let $(G_n)_n$ be a sequence of d-regular graphs such that $\lim_{n\to\infty} g(G_n) = \infty$.
Then

$$\lim_{n\to\infty} F(G_n)^{1/v(G_n)} = \lim_{n\to\infty} a(G_n)^{1/v(G_n)} = \frac{(d-1)^{d-1}}{(d^2-2d-1)^{d/2-1}}.$$

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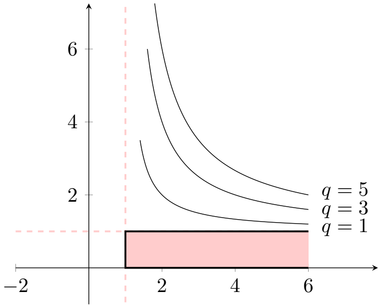
Related results

Random cluster model:

$$Z_G(q,w) = \sum_{A \subseteq E(G)} q^{k(A)} w^{|A|}.$$

Then $q = (x-1)(y-1)$ and $w = y - 1$.



Dembo, Montanari, Sun +Sly:	$q > 1$ **integer**, $w \geq 0$,
Helmuth, Jenssen and Perkins:	q **large**, $w \geq 0$ and Expansion.
Galanis, Štefankovič, Vigoda and Yang:	$q > 3$, $w \geq 0$ and random d-regular graphs

Extremal questions

What is the supremum of $\tau(G)^{1/n}$ for d-regular graphs?

Theorem (McKay)

Let G be a d-regular graph on n vertices, then

$$\tau(G) \leq \frac{c_d \log n}{n} \left(\frac{(d-1)^{d-1}}{(d^2-2d)^{d/2-1}} \right)^n$$

Extremality of Forests?

Let G be a d-regular graph on n vertices.

Thomassen:	$F(G) \leq (d+1)^n$
Kahale-Schulman:	$F(G) \leq C_d^n = \left(d + \frac{1}{2} + O(1/d)\right)^n$
Borbényi-Csikvári-Luo:	$F(G) \leq \hat{C}_d^n < C_d^n$ for $d \in \{3, 4, \ldots, 9\}$
Bencs-Csikvári:	$F(G) < d^n$

Conjecture (Bencs and Csikvári)

Let G be a d-regular graph G on n vertices. Let $F(G)$ denote the number of spanning forests of G. Then

$$F(G) \leq \left(\frac{(d-1)^{d-1}}{(d^2 - 2d - 1)^{d/2-1}} \right)^n .$$

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Behind the curtains!

Plan for $T_G(x,1)$

Let

$$F_G(z) := z^{k(G)} T_G(z+1,1) = z^{k(G)} \sum_{A \subseteq E} z^{k(A)-k(G)} (0)^{k(A)+|A|-v(G)}$$

Then

$$F_G(z) = \sum_{k=0}^{n} f_k(G) z^{n-k},$$

where $f_k(G)$ denotes the number of forests with exactly k edges.

The polynomial $R_G(z)$

Let

$$R_G(z) = \sum_{M \in \mathcal{M}(G)} (-z)^{|M|} \prod_{v \notin V(M)} (z + d_v - 1),$$

where d_v is the degree of the vertex v, and $\mathcal{M}(G)$ is the set of matchings of G including the emptyset.

Comparison

Theorem

Let G be a graph on n vertices with average degree $\bar{\mathrm{d}}$ such that it contains at most L cycles of length at most $g-1$. Then

$$\left(1+\frac{g\bar{d}}{z}\right)^{-n/g} R_G(z+1) \leq F_G(z) \leq R_G(z+1).$$

Corollary

If $g(G_n) \to \infty$ and the average degree is constant, then for any $z > 1$ we have

$$\lim F_G^{1/n}(z) = \lim R_G^{1/n}(z+1)$$

Relation to Mathing polynomials

Let

$$\mu_G(z) = \sum_{k=0}^{n/2} (-1)^k m_k(G) z^{n-2k},$$

where $m_k(G)$ denotes the number of matchings of size k.

Note that for a d-regular graph G we have

$$R_G(z) = \sum_{M \in \mathcal{M}(G)} (-z)^{|M|} (d+z-1)^{n-2|M|} = z^{n/2} \mu_G\left(\frac{d-1+z}{\sqrt{z}}\right).$$

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Matching polynomial

Theorem (Heilmann and Lieb)

All zeros of the matching polynomial $\mu_G(z)$ are real. Furthermore, if the largest degree Δ satisfies $\Delta \geq 2$, then all zeros lie in the interval $(-2\sqrt{\Delta - 1}, 2\sqrt{\Delta - 1})$.

Summary

$$T_G(z+1,1)^{1/v(G)} \approx F_G(z)^{1/v(G)} \approx R_G(z+1)^{1/v(G)}$$
$$= (z+1)^{1/2}\left(\mu_G\left(\frac{d-1+(z+1)}{\sqrt{z+1}}\right)\right)^{1/v(G)}.$$

where $\mu_G(z) = \prod_{i=1}^{v(G)}(z-\alpha_i)$ and

$$\left(\mu_G\left(\frac{d-1+z}{\sqrt{z}}\right)\right)^{1/v(G)} = \left(\prod_{i=1}^{v(G)}\left(\frac{d-1+z}{\sqrt{z}}-\alpha_i\right)\right)^{1/v(G)} =$$
$$= \exp\left(\frac{1}{v(G)}\sum_{i=1}^{v(G)}\ln\left(\frac{d-1+z}{\sqrt{z}}-\alpha_i\right)\right)$$
$$\approx \exp\left(\int \ln\left(\frac{d+z-1}{\sqrt{z}}-t\right)d\rho_{\mathrm{KM}}(t)\right).$$

Plan for $T_G(x, 0)$

We show that $T_G(x, 0)$ is not far from $T_G(x, 1)$ for large girth graphs. "Not far" means subexponential ratio. In fact, even their coefficients are not far in this sense.

Thank you very much for your attention!

Broken cycles

Definition

Given a graph G and an ordering of the edges, a path is called a broken cycle if it can be obtained from a cycle by deleting the edge with highest index.

Lemma

Let G be a connected graph on n vertices with an arbitrary ordering of the edges. Let

$$z^{k(G)}T_G(z+1,0) = \sum_{k=1}^{n} c_k z^{n-k}.$$

Then c_k counts the number of edge sets of size k containing no broken cycle.

Comparison of $T_G(x,0)$ and $T_G(x,1)$

$$z^{k(G)}T_G(z+1,0) = \sum_{k=1}^{n} c_k z^{n-k} \quad \text{and} \quad z^{k(G)}T_G(z+1,1) = \sum_{k=1}^{n} f_k z^{n-k}.$$

Lemma

Let G be a graph with n vertices, m edges and at most L cycles of length at most $g-1$. Let c_k be the number of edge sets with exactly k edges with no broken-cycle, and let f_k be the number of forests with exactly k edges. Then

$$c_k \geq \left(\frac{2}{3}\right)^L \left(1 - \frac{1}{g}\right)^{m-n+k(G)-L} f_k.$$

Proof idea

$$c_k \geq \left(1 - \frac{1}{g}\right)^{m-n+k(G)} f_k.$$

Idea of the proof:

- c_k is independent of the order π of the edges, so take a random π.
- For a fixed forest estimate the probability that it remains broken cycle-free.
- Observation: in the above step one can use positive correlation.
- Finally, use linearity of expectation.