Extremal entropy of graphs

based on joint work with Matteo Mazzamurro and Yanni Dong

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What is entropy?

It is a term used in

- thermodynamics
- information theory
- statistical physics
- chemistry, biology

What is entropy?

It is associated with

- disorder
- randomness
- uncertainties and probabilities

Which entropy we are dealing with?

Definition

The Shannon entropy associated with a finite probability distribution $\overrightarrow{p} = (p_i)_i$ *equals* $H(\overrightarrow{p}) = -\sum_i p_i \log(p_i)$.

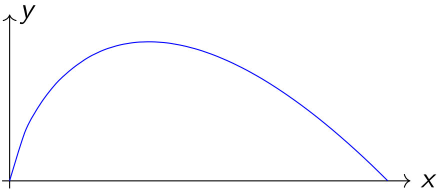


Figure: Plot of $g(x) = -x \log(x)$ for $x \in [0, 1]$

Which entropy we are dealing with?

Definition

The Shannon entropy associated with a finite probability distribution $\overrightarrow{p} = (p_i)_i$ *equals* $H(\overrightarrow{p}) = -\sum_i p_i \log(p_i)$.

For graphs, we may choose the probabilities over the vertices linearly with its degree.

Which entropy we are dealing with?

Definition

The (first degree-based) entropy is the Shannon entropy associated with its normalized degree-sequence.

$$I(G) = -\sum_{i=1}^{n} \frac{d_i}{2m} \log\left(\frac{d_i}{2m}\right).$$

The entropy of graphs

Definition (entropy of graphs)

$$
\begin{aligned}
I(G) &= -\sum_{i=1}^{n} \frac{d_i}{2m} \log\left(\frac{d_i}{2m}\right) \\
&= \log(2m) - \frac{1}{2m} h(G).
\end{aligned}
$$

where

$$
h(G) = \sum_i f(d_i) = \sum_i d_i \log(d_i)
$$

What can we tell about $I(G)$?

Definition

$$I(G) = -\sum_{i=1}^{n} \frac{d_i}{2m} \log\left(\frac{d_i}{2m}\right).$$

Remark

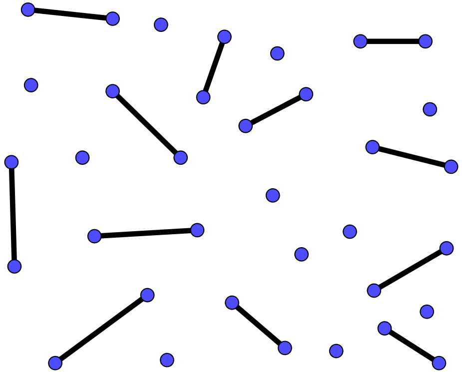
We can only tell something about $I(G)$ if we know the extrema.

What graphs would be extremal given size?

Question

Which graph would maximize the entropy among all graphs with size m?

What graphs would be extremal given size?



$$h(G) = h(mK_2) = 0$$

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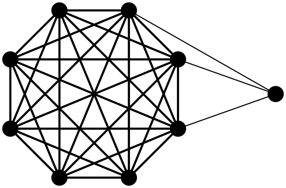

Figure: The graph $\mathcal{C}(31)$

Theorem

Among all graphs with size m, the colex graph $\mathcal{C}(m)$ maximizes $h(G)$ and therefore minimizes the entropy.

What graphs would be extremal given size and order?

Question

What graphs with order n and size m would maximize the entropy?

What graphs would be extremal given size and order?

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What graphs with order n and size m would maximize the entropy?

The graph $G_{n,m}$ could be a first guess, but not completely true.

What graphs would be extremal given size and order?

Question

What graphs with order n and size m would maximize the entropy?

Answer

Nearly regular graphs, i.e. $\Delta - \delta \leq 1$.
Reason: $f(x) = x\log(x)$ *is non-decreasing and strictly convex for* $x \geq 1$ *and so minimum* $h(G)$ *when degree sequence balanced*

What graphs would be extremal given size and order?

Question

What connected graphs with order n and size m would minimize the entropy?

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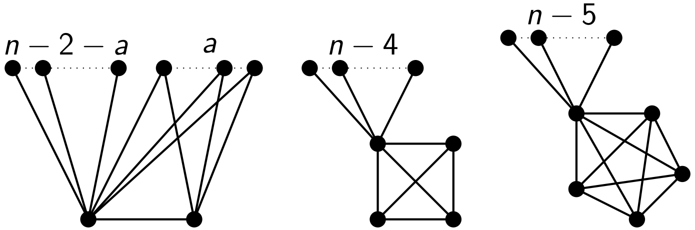
Question

What connected graphs with order n and size m would minimize the entropy?



What about bipartite graphs?

Question

Which graph minimizes the entropy among all bipartite graphs with given order n and size m?

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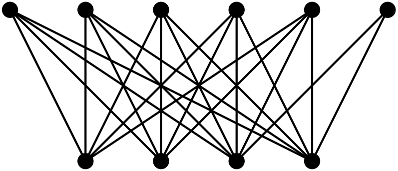
Answer

Every complete bipartite $K_{q,y}$ with $qy = m$ and $q + y \leq n$, or a graph that is nearly complete bipartite.

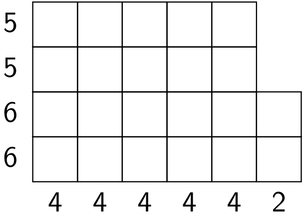
What about bipartite graphs?

Question

Which graph minimizes the entropy among all bipartite graphs with given order n and size m?



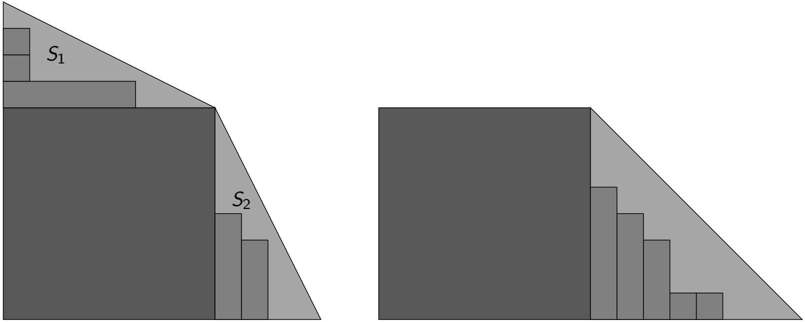
(a) Graph representation



(b) Associated Young tableau

Figure: Two representations of the extremal bipartite $(10, 22)$-graph $B(10, 22, 4)$

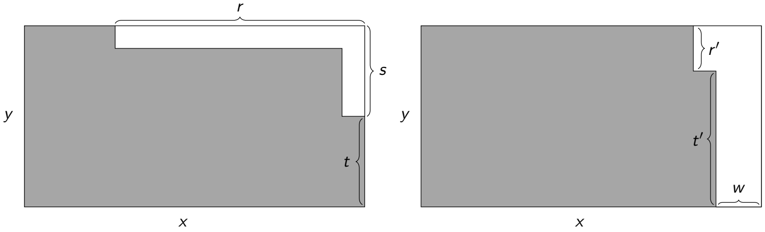
Increasing h by rearranging T



(a) Young tableau T (b) Young tableau T'

Figure: Sketch of a rearrangement

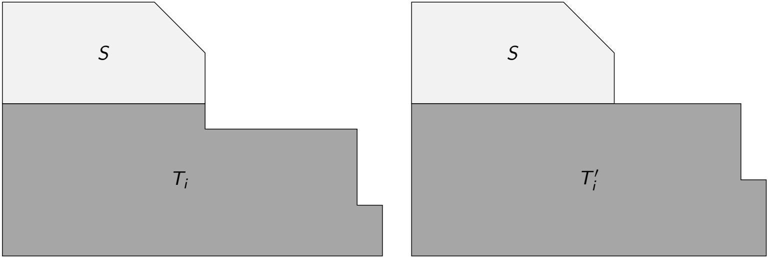
Increasing h by rearranging T



(a) Young tableau T (b) Young tableau T'

Figure: Another improvement of a tableau

Iteratively improving



(a) Young tableau T (b) Young tableau T'

Figure: The local move increasing $h(T)$

The end

Thank you for your attention!

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