Random colourings of trees with constant down degree

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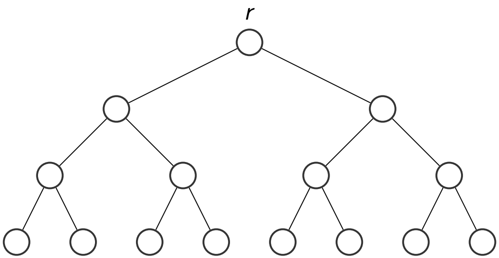
Random proper colouring



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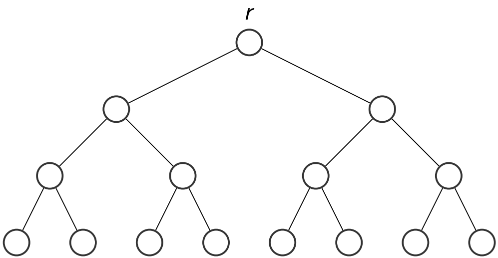
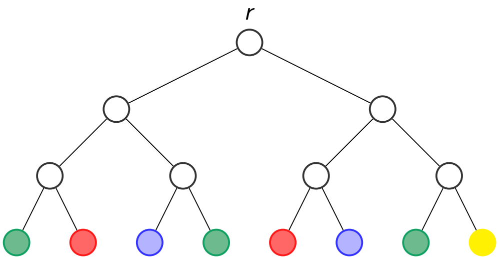


Question

What is the probability for a random proper colouring that the root r gets colour 1?

Boundary condition



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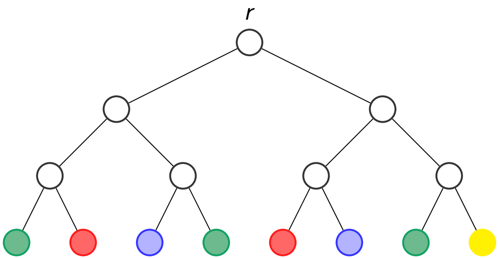
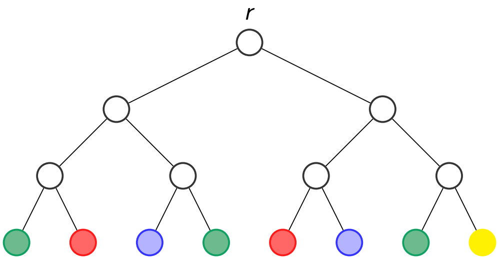


Question

What is the probability for a random proper colouring that the root r gets colour blue, conditioned on the given boundary condition?

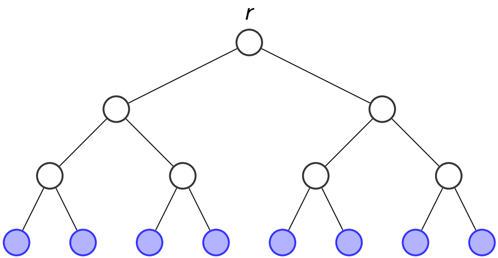
Boundary condition



Question

What is the probability for a random proper colouring that the root r gets colour blue, conditioned on the given boundary condition? More specifically, what happens with these probabilities as we let the distance n between leaves and root grow and look at all possible boundary conditions? Does it go to $1/q$?

Random proper colouring $q = 2$



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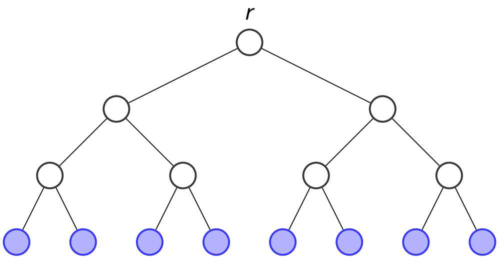
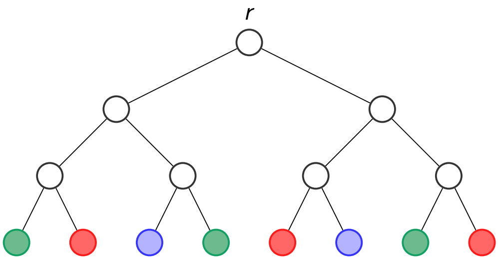


Remark

Here we see the probability that the roots gets colour blue is 1.

Boundary condition $q = 3$



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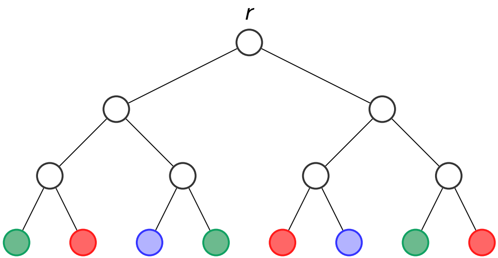

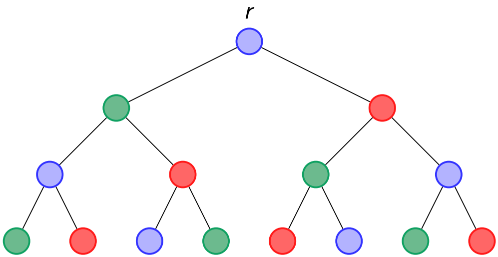

Jonasson

Theorem (Jonasson 2002)

Let $q > d + 1$. Then the probability that the roots gets colour i for a random proper colouring tends to $1/q$ as the distance between the root and the leaves goes to infinity, independent of the boundary condition on the leaves.

Potts model

Definition

Let $G = (V, E)$ a finite graph, let $q \in \mathbb{N}_{\geq 2}$ the number of colours and let $w \geq 0$ be an interaction parameter. A colouring $\sigma : V \to [q] := \{1, \dots, q\}$ gets associated weight

$$W_G(\sigma) := w^{\#\text{ monochromatic edges in } \sigma}.$$

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Remark

Select colourings ϕ of the tree with probability proportional to $W(\phi)$. We will look at what happens with

$$\mathbf{P}[\text{ root has color } i \text{ given the leaves are colored with } \tau]$$

as the distance n between the root and the leaves grows to ∞.

The conjecture and our main result

Folklore conjecture

Denote $w_c = 1 - \frac{q}{d+1}$. The probability vector of the root of the tree with down degree d goes to the vector $(1/q, \ldots, 1/q)$ if and only if w is

$$\begin{cases} \geq 0 & \text{for } d+1 < q \\ > 0 & \text{for } d+1 = q \\ \geq w_c & \text{for } d+1 > q \end{cases}$$

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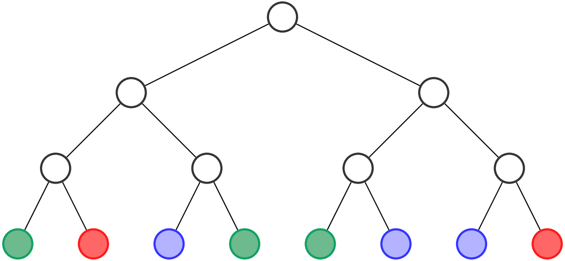
Theorem

For each $q \geq 3$ there is a d_0 big enough such that for each $d \geq d_0$ the q-state anti-ferromagnetic Potts model on $\mathbb{T}_d$ has the "uniform probabilities property" if and only if $w \geq w_c = 1 - \frac{q}{d+1}$.

Step 1: using ratios



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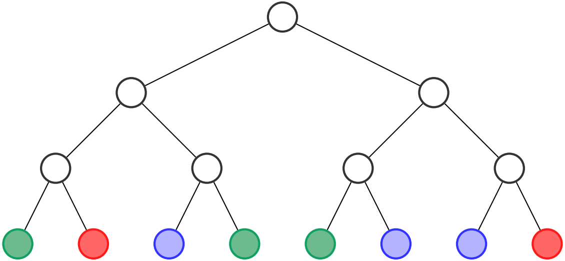
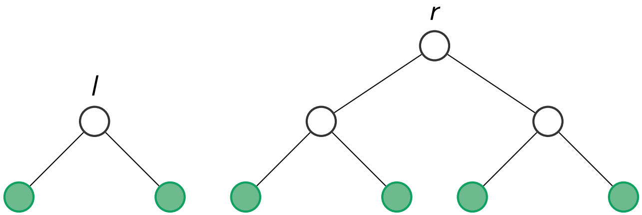


Definition

For every color $i \in [q]$ and every boundary condition τ on the leaves we define the ratio as

$$R_{\tau,i} = \frac{\mathbf{P}[\text{ root has color } i \text{ given the leaves are colored with } \tau]}{\mathbf{P}[\text{ root has color } q \text{ given the leaves are colored with } \tau]}.$$

Step 2: using a recursion



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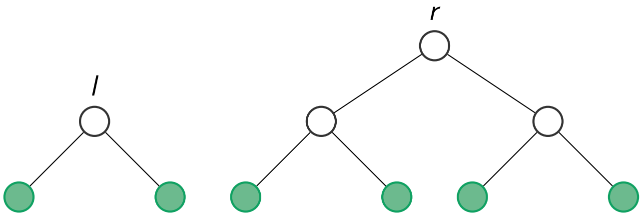
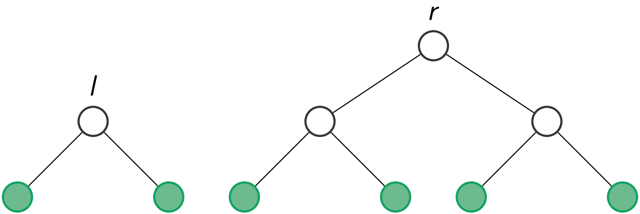


Example

l: $(R_{\bullet}, R_{\bullet}) =$

Step 2: using a recursion



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l: $(R_{\bullet}, R_{\bullet}) = (\frac{1}{w^2}, \frac{1}{w^2})$.

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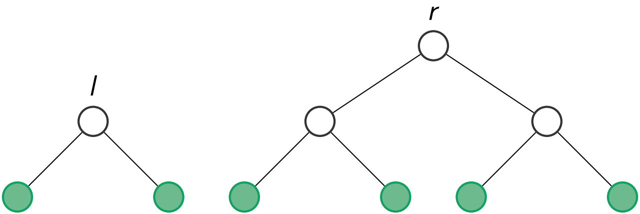

Example

l: $(R_{\color{red}\bullet}, R_{\color{gray}\bullet}) = (\frac{1}{w^2}, \frac{1}{w^2})$. r: $(R_{\color{red}\bullet}, R_{\color{gray}\bullet}) = ((\frac{w^2+w+1}{w^3+1})^2, (\frac{w^2+w+1}{w^3+1})^2)$.

Definition

We define F as

$$F(R_{\color{red}\bullet}, R_{\color{gray}\bullet}) = \left(\left(\frac{wR_{\color{red}\bullet} + R_{\color{gray}\bullet} + 1}{R_{\color{red}\bullet} + R_{\color{gray}\bullet} + w}\right)^2, \left(\frac{R_{\color{red}\bullet} + wR_{\color{gray}\bullet} + 1}{R_{\color{red}\bullet} + R_{\color{gray}\bullet} + w}\right)^2\right).$$

Step 3: a geometric condition

Remark

In general we have that new ratios are of the form

$$F(R_{\color{red}\bullet,1}, R_{\color{gray}\bullet,1})^{1/d} \cdots F(R_{\color{red}\bullet,d}, R_{\color{gray}\bullet,d})^{1/d}.$$

Step 3: a geometric condition

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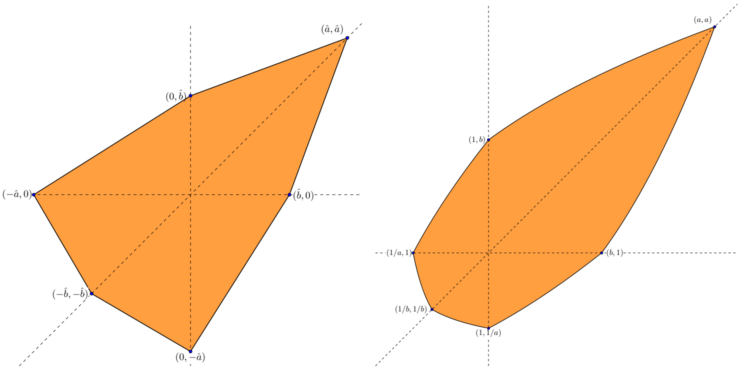

Figure: Using log-convex sets

Step 4: using a limit

Definition

In logarithmic coordinates we define (for $q = 3$) the map $\hat{F}(\hat{R}_{\color{red}\bullet}, \hat{R}_{\color{gray}\bullet})$ as

$$\left(d \log\left(1 + \frac{\alpha q}{d+1} \cdot \frac{1 - e^{\hat{R}_{\color{gray}\bullet}}}{e^{\hat{R}_{\color{red}\bullet}} + e^{\hat{R}_{\color{gray}\bullet}} + w}\right), d \log\left(1 + \frac{\alpha q}{d+1} \cdot \frac{1 - e^{\hat{R}_{\color{red}\bullet}}}{e^{\hat{R}_{\color{red}\bullet}} + e^{\hat{R}_{\color{gray}\bullet}} + w}\right)\right),$$

where $w = 1 - \alpha\frac{q}{d+1}$ for $\alpha \in (0, 1]$.

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where $w = 1 - \alpha \frac{q}{d+1}$ for $\alpha \in (0, 1]$.

Definition

Get the limit map $\hat{F}_{\infty}(\hat{R}_{\textcolor{red}{\bullet}}, \hat{R}_{\textcolor{gray}{\bullet}})$

$$\left(\alpha q \cdot \frac{1 - e^{\hat{R}_{\textcolor{gray}{\bullet}}}}{e^{\hat{R}_{\textcolor{red}{\bullet}}} + e^{\hat{R}_{\textcolor{gray}{\bullet}}} + 1}, \alpha q \cdot \frac{1 - e^{\hat{R}_{\textcolor{red}{\bullet}}}}{e^{\hat{R}_{\textcolor{red}{\bullet}}} + e^{\hat{R}_{\textcolor{gray}{\bullet}}} + 1}\right).$$

Analyse this map for $\alpha = 1$, so $w = w_c$, then use the uniform convergence of the finite maps to get a result for large d

The end

Thank you for your attention!