Dutch Day of Combinatorics

Higgledy-piggledy sets
in projective spaces

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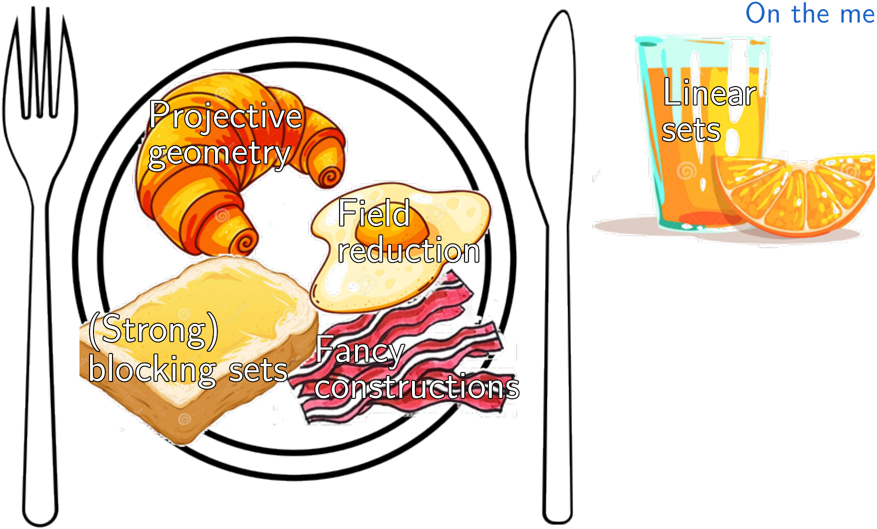
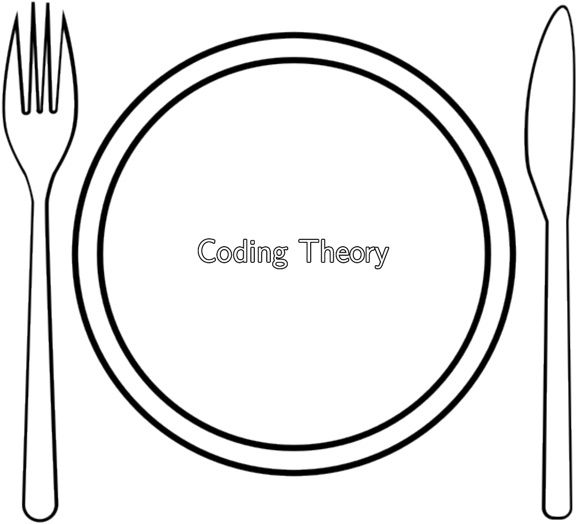
Overview

On the menu


Projective
geometry
Field
reduction
(Strong)
blocking sets
Fancy
constructions
Linear
sets

On the menu

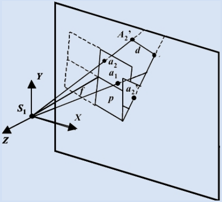


Crash course in projective geometry

Let $N \in \mathbb{N}^\times$ and q be a prime power. Assume $\mathrm{V}(N+1, q)$ to be the $(N+1)$-dimensional vector space over $\mathbb{F}_q$.

The Desarguesian projective geometry $\mathrm{PG}(N, q)$ is the incidence geometry of subspaces of $\mathrm{V}(N+1, q)$, i.e.
$\mathrm{V}(N+1, q) \cong \mathrm{PG}(N, q)$.

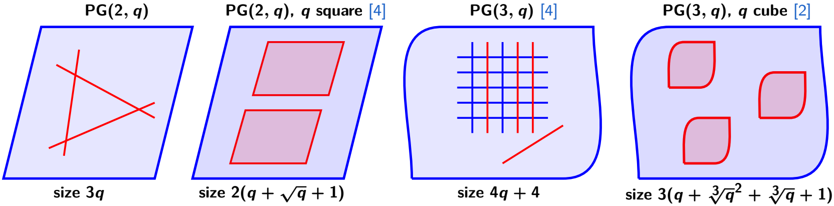
- vector lines are (proj.) points.
- vector planes are (proj.) lines.
- vector solids are (proj.) planes.
- ...



All dimensions lower by 1, but Grassmann's identity still holds!

Let $k \in \{0, \ldots, N-1\}$.

A *strong k-blocking set* is a point set that meets every $(N-k)$-dimensional subspace κ in a point set spanning κ.



A *higgledy-piggledy set of k-subspaces* is a set $\mathcal{K}$ of k-subspaces such that the point set $\cup\mathcal{K}$ is a strong k-blocking set.

Motivation

Let $n \in \mathbb{N}^{\times}$ and $k' \in \{0, 1, \ldots, n\}$.

A (q-ary) *linear* $[n, k']_q$*-code* $\mathcal{C}$ is a k'-dim. subspace of $\mathrm{V}(n, q)$.

- Vectors in $\mathcal{C}$ are called *codewords*.
- The *support* $\mathrm{supp}(c)$ of a codeword $c \in \mathcal{C}$ is equal to the subset of $\{1, 2, \ldots, n\}$ of non-zero positions in c.

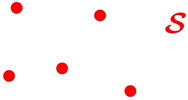
A codeword $c \in \mathcal{C}$ is called *minimal* if

$$\forall c' \in \mathcal{C} : \mathrm{supp}(c') \subseteq \mathrm{supp}(c) \Rightarrow c' \in \langle c \rangle_{\mathbb{F}_q}.$$

A linear code is called *minimal* if all its codewords are minimal.

2 Motivation

Correspondence with minimal codes



strong block. set w.r.t. hyperplanes, $\mathcal{S} := \{P_1, \ldots, P_{|\mathcal{S}|}\}$.

$$\begin{array}{ccccccc} P_1 & P_2 & P_3 & & P_i & & P_{|\mathcal{S}|} \\ \end{array}$$

$$\begin{pmatrix} x_{10} & x_{20} & x_{30} & \cdots & x_{i0} & \cdots & x_{|\mathcal{S}|0} \\ x_{11} & x_{21} & x_{31} & \cdots & x_{i1} & \cdots & x_{|\mathcal{S}|1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1N} & x_{2N} & x_{3N} & \cdots & x_{iN} & \cdots & x_{|\mathcal{S}|N} \end{pmatrix}$$

coordinates of P_i

Theorem ([1, 8]).
$\rightarrow$ the generator matrix of a **minimal** linear $[|\mathcal{S}|, N+1]_q$-code!

Goal: Finding small ~~strong blocking sets~~ higgledy-piggledy sets.

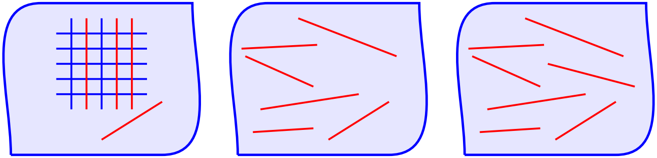
3 Known results

The line case $(k = 1)$

Theorem ([5, 7]).
If $q \geqslant N + \lfloor \frac{N}{2} \rfloor$, then
A higg.-pigg. line set contains at least $N + \lfloor \frac{N}{2} \rfloor - \lfloor \frac{N-1}{q} \rfloor$ lines.

Theorem ([5, 7]).
If $q \geqslant 2N - 1$, then
There exists a higg.-pigg. line set of asymptotic size $2N - 1$.

PG(3, q) [4] — **PG(4, q)**, $q > 36086$, $\mathrm{char}(q) \neq 2, 3$ [3] — **PG(5, q)** [2]



size $4q + 4$ — **size $6q + 6$** — **size $7q + 7$**

Construction methods

Theorem ([5]).
A line set $\mathcal{K}$, $|\mathcal{K}| \leqslant q$, is a higg.-pigg. set
$\Leftrightarrow$ no $(N-2)$-subspace meets all its lines.

Projection
eg. consider one parity case.

Dualisation
$k \to N - k - 1$.

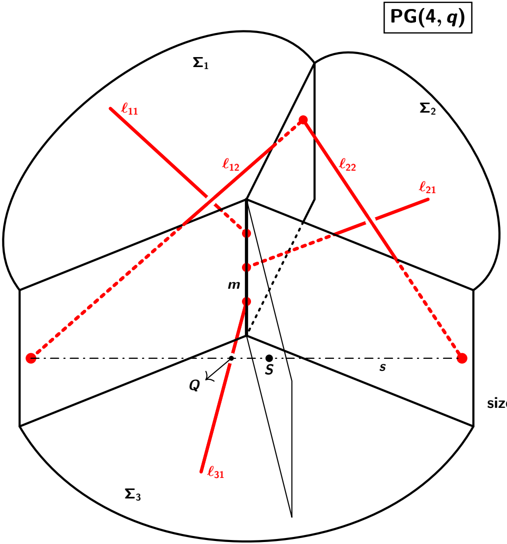
Field reduction
if $N+1$ is composite.

Coordinates
example in $\mathrm{PG}(4, q)$.

Probability
improvements on lower/upper bounds.

Elem. geometry
see next slide.

An example



size $6q + 5$

Conclusion

- Lower the existence of $2N-1$ higg.-pigg. lines to $\lfloor \frac{3}{2}N \rfloor$ higg.-pigg. lines?
- Lower the existence of $(N-k)(k+1)+1$ higg.-pigg. k-spaces [6, 7]?
- Exploit the known classification results on linear sets to find small higg.-pigg. sets?
- Improve current probabilistic methods?

L. Denaux. Higgledy-piggledy sets in projective spaces of small dimension. *arXiv e-prints*, Sep 2021. arXiv:2109.08572

Thank you for listening

Any questions?

Suggestions?

Marvellous revelations?

References

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