Graph reconstruction from partial information

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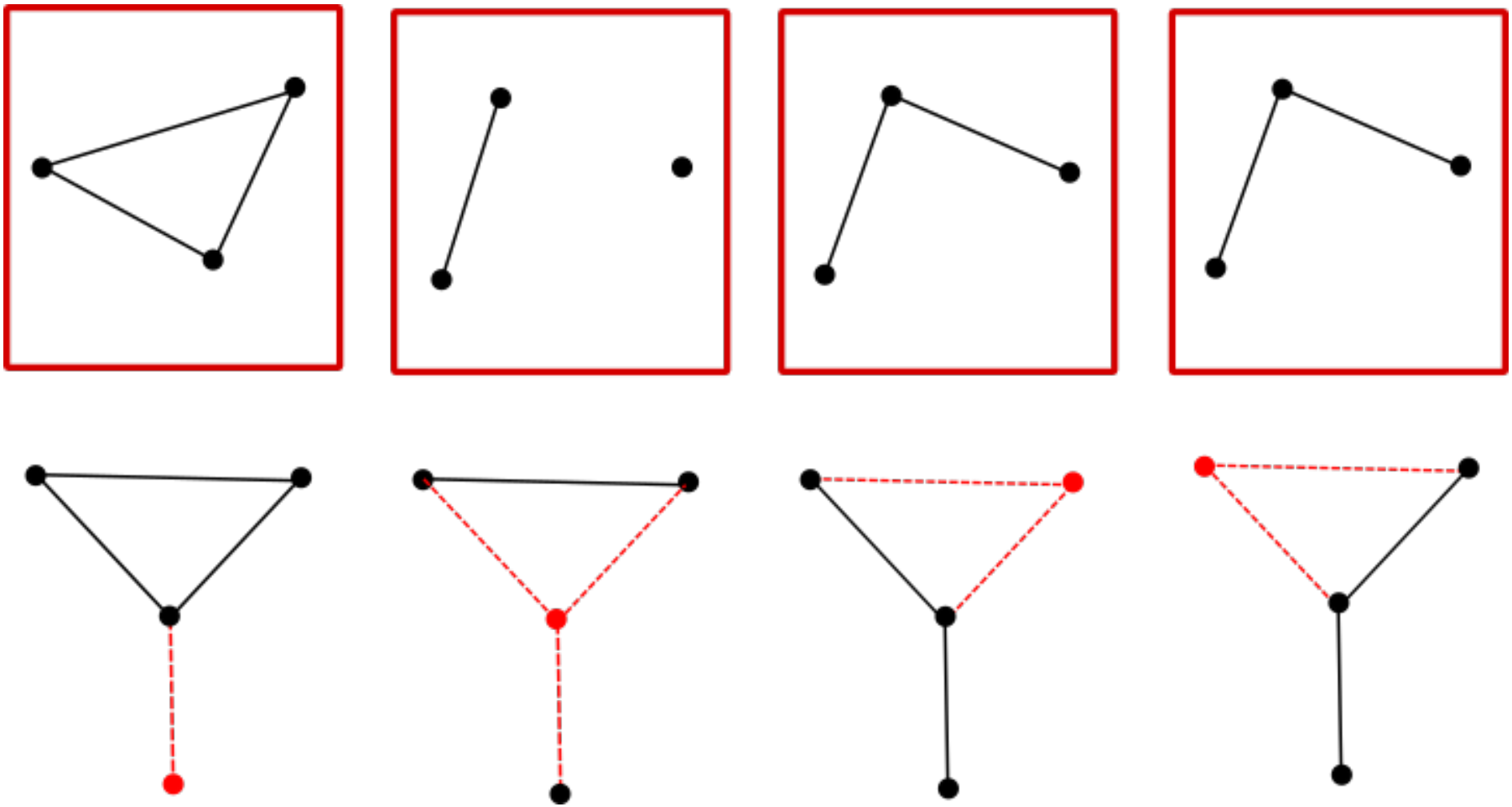
Utrecht University

Joint work with Tom Johnston, Alex Scott and Jane Tan

Thursday May 12, 2022

Wine glasses are reconstructible

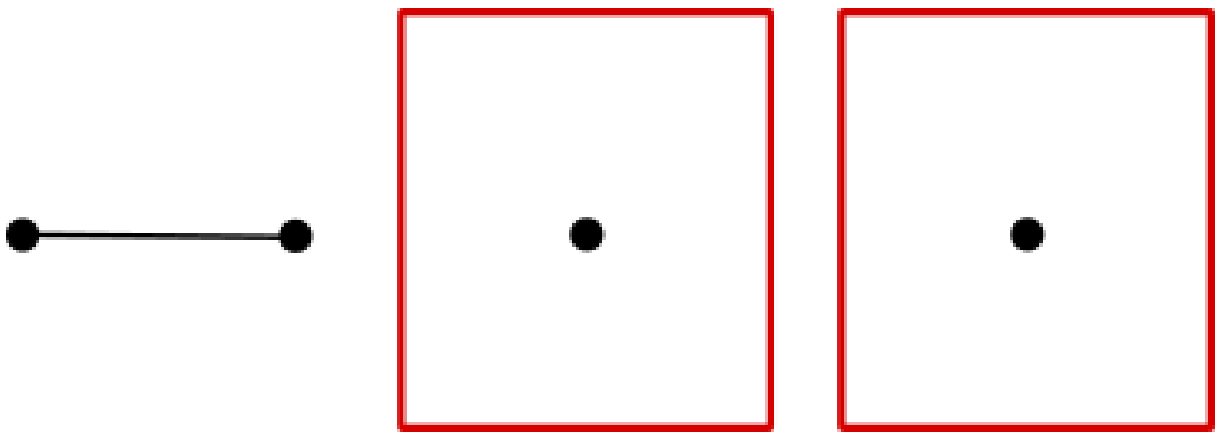


Graph reconstruction

"Is a graph determined by its strict induced subgraphs?"

- Unlabelled, simple graph G on n vertices.
- Card: $G - v$ induced subgraph on $n - 1$ vertices ($v \in V(G)$).
- Deck: $\{G - v : v \in V(G)\}$ multiset of cards.
- G is *reconstructible* if every graph with the same deck is isomorphic to it.

Single edge is not reconstructible



Graph reconstruction conjecture

G is *reconstructible* if every graph with the same deck is isomorphic to it.

Reconstruction conjecture (Kelly and Ulam)

Every graph G on $n \geq 3$ vertices is reconstructible.

P. Kelly. *On isometric transformations*. PhD thesis, University of Wisconsin, 1942.

S. Ulam. *A collection of mathematical problems*, volume 8. Interscience Publishers, 1960.

A warning from Bondy and Hemminger

"The Reconstruction Conjecture is generally regarded as one of the foremost unsolved problems in graph theory. Indeed, Harary (1969) has even classified it as a 'graphical disease' because of its contagious nature.

According to reliable sources, it was discovered in Wisconsin in 1941 by Kelly and Ulam, and claimed its first victim (P. J. Kelly) in 1942.

There are now more than sixty recorded cases, and relapses occur frequently (this article being a case in point)."

J. Bondy and R. Hemminger. Graph reconstruction – a survey, *Journal of Graph Theory* **1**:227-268, 1977.

So what is known?

- Trees, maximal planar graphs, disconnected graphs and regular graphs are reconstructible.
- $G(n, p)$ is almost surely reconstructible (from any 3 cards for p fixed!)

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- Trees, maximal planar graphs, disconnected graphs and regular graphs are reconstructible.
- $G(n, p)$ is almost surely reconstructible (from any 3 cards for p fixed!)
- Edge reconstruction conjecture also still open.
- Digraph version false.[1]
- Infinite version false.[2]

[1] P. Stockmeyer. The falsity of the reconstruction conjecture for tournaments. *Journal of Graph Theory* **1**:19–25, 1977.

[2] N. Bowler, J. Erde, P. Heinig, F. Lehner and M. Pitz. A counterexample to the reconstruction conjecture for locally finite trees. *Bulletin of the London Mathematical Society*, 2017.

Reconstructing graph parameters

A graph parameter is *reconstructible* if its value only depends on the deck.

- Number of edges $e(G)$ of G from

$$\sum_{v \in V(G)} e(G - v) = (n-2)e(G)$$

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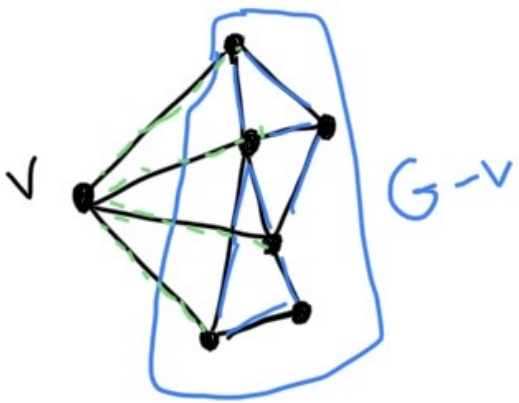

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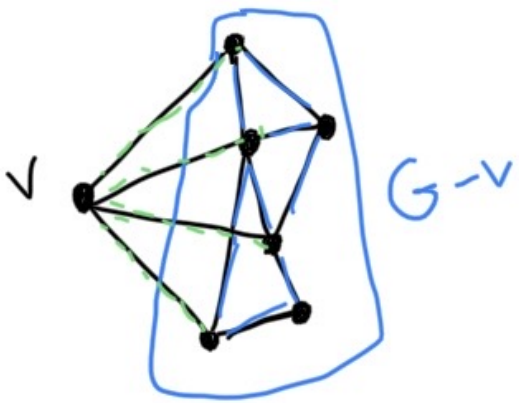

- Degree sequence from

$$d(v) = e(G) - e(G - v)$$

- Subgraph counts, connectivity, planarity, chromatic polynomial, ...

Reconstruction from partial information

$\mathcal{D}(G) = \mathcal{D}_{n-1}(G) =$ multiset of the n induced subgraphs on $n-1$ vertices.

Restrictions on the decks:

- Partial deck: less than n cards are given.
- Small cards: order of the subgraphs on the cards is smaller.

Definition of ℓ-deck

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Degree reconstruction from small cards

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For which ℓ can the degree sequence of any n-vertex graph G be reconstructed from $\mathcal{D}_\ell(G)$?

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Kostochka, Nahvi, West and Zirlin[1]: $\ell \geq n-3$ for $n \geq 7$.

Taylor[2]: for n sufficiently large, $\ell \geq \left(1-\frac{1}{e}\right) n$.

[1] A. Kostochka, M. Nahvi, D. West and D. Zirlin. Degree lists and connectedness are 3-reconstructible for graphs with at least seven vertices. *Graphs Combin.*, **36**:491– 501, 2020.

[2] R. Taylor. Reconstructing degree sequences from k-vertex-deleted subgraphs. *Discrete Math.*, **79**(2):207–213, 1990.

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Theorem (G., Johnston, Scott, Tan)

For $n \geq 3$, $\ell \geq \sqrt{2n \log(2n)}$.

Recognising connectedness from small cards

Kostochka, Nahvi, West and Zirlin[1]: connectedness can be recognised from $\mathcal{D}_{n-3}(G)$ for $n \geq 7$.

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$$\ell \geq n - (1 + o(1))\sqrt{\frac{2\log(n)}{\log(\log(n))}}.$$

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Moreover, $P_{\lfloor n/2 \rfloor - 1} \sqcup C_{\lceil n/2 \rceil + 1}$ and P_n have the same $\lfloor n/2 \rfloor$-deck.

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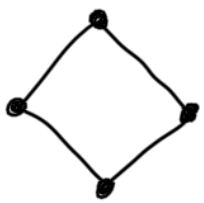
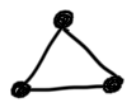
$n=6$



$C_4 \cup P_2$

P_6

cards	$C_4 \cup P_2$	P_6
		

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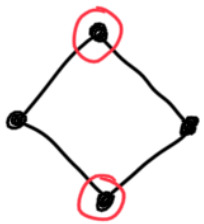

$C_4 H P_2$

P_6

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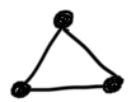		

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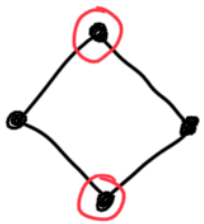

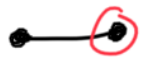

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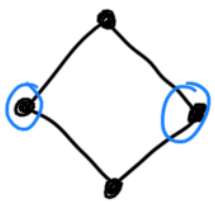

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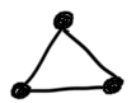		

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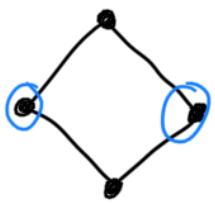

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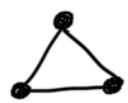		

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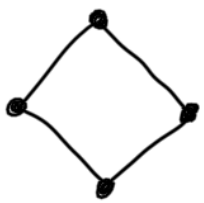

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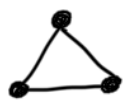
cards	$C_4 H P_2$	P_6
	4	
		

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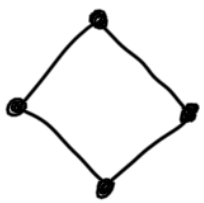

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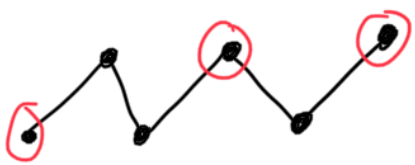

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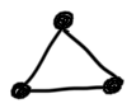		

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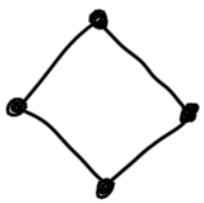

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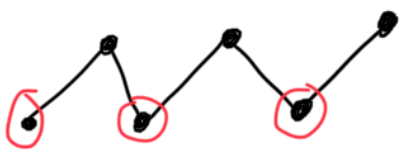

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cards	$C_4 \cup P_2$	P_6
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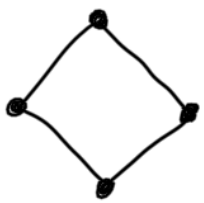

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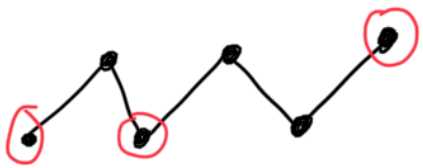
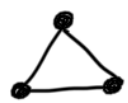


P_6

cards	C_4 H P_2	P_6
	4	
		

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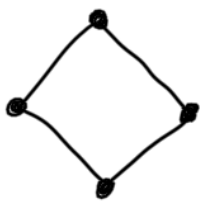

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P_6

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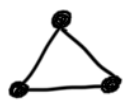
	4	4
	0	0

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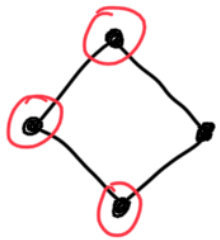

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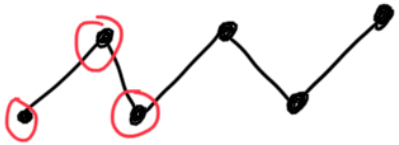
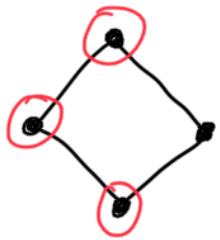


P_6

cards	$C_4 \sqcup P_2$	P_6
	4	4
	rest	rest
	4	4
	0	0

$n=6$



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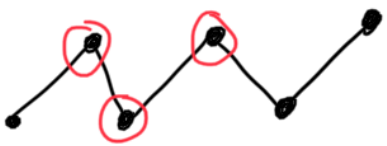

P_6

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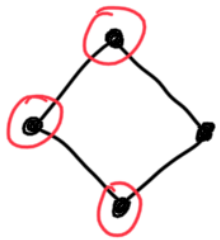
	4	4
	rest	rest
	4	4
	0	0

$n=6$



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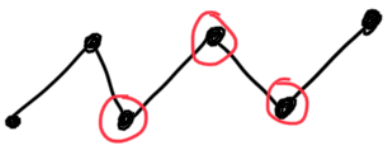

P_6

cards	$C_4 H P_2$	P_6
	4	4
	rest	rest
	4	4
	0	0

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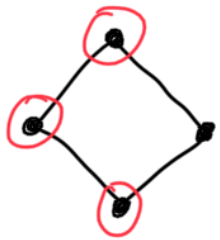

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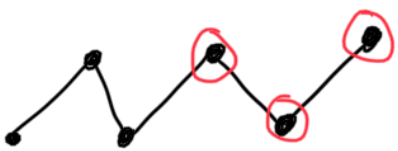

P_6

cards	$C_4 H P_2$	P_6
	4	4
	rest	rest
	4	4
	0	0

Recognising connectedness from small cards

Theorem (G., Johnston, Scott, Tan)

For $n \geq 3$, the connectedness of an n-vertex graph G can reconstructed from $\mathcal{D}_\ell(G)$ for any $\ell \geq 0.9n$.

Tight up to constant: disconnected G and connected H may have the same $\lfloor n/2 \rfloor$-deck.

Previous best asymptotic bound: $n - \sqrt{\frac{2\log(n)}{\log(\log(n))}}$.

Theorem (Borwein, Erdélyi and Kós)

Any complex polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ has at most $\sqrt{2n \log\left(\frac{|a_0|+|a_1|+\cdots+|a_n|}{\sqrt{|a_0 a_n|}}\right)}$ positive real roots.

P. Borwein, T. Erdélyi, and G. Kós. Littlewood-type problems on $[0, 1]$. *Proc. Lond. Math. Soc.*, **79**(1):22–46, 1999.

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Main Lemma (Borwein and Ingalls)

Let $\alpha, \beta \in \{0, \ldots, n\}^m$ not related by permutation. If

$$\binom{\alpha_1}{j} + \cdots + \binom{\alpha_m}{j} = \binom{\beta_1}{j} + \cdots + \binom{\beta_m}{j} \text{ for all } j \in \{0, \ldots, r\}$$

then $r + 1 \leq \sqrt{2n \log(2m)}$.

P. Borwein, T. Erdélyi, and G. Kós. Littlewood-type problems on $[0, 1]$. *Proc. Lond. Math. Soc.*, **79**(1):22–46, 1999.

P. Borwein and C. Ingalls, The Prouhet-Tarry-Escott problem revisited, *Enseign. Math.*, **40**(2):3–27, 1994.

Degree sequence: objective

Definition of ℓ-deck

$\mathcal{D}_\ell(G)$ is the multiset of all induced subgraphs of order ℓ.

For $n \geq 3$ and $\ell \geq \sqrt{2n \log(2n)}$, for pairs of n-vertex graphs

$$\text{same } \ell\text{-deck} \implies \text{same degree sequence.}$$

Degree sequence: reconstructing star-counts

Suppose we obtain the ℓ-deck of G.

Special case of Kelly's lemma

For all $j+1 \leq \ell$, we may reconstruct the number of subgraphs $n_{K_{1,j}}(G)$ of G isomorphic to $K_{1,j}$.

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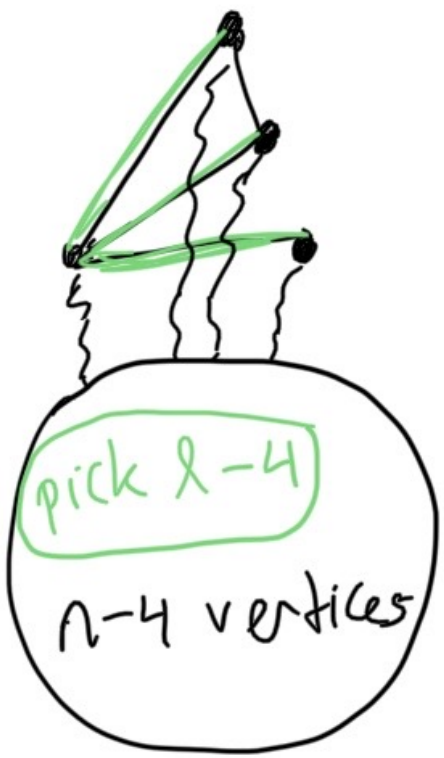
Counting shows $\sum_{C \in \mathcal{D}_\ell(G)} n_{K_{1,3}}(C) = n_{K_{1,3}}(G)\binom{n-4}{\ell-4}$.



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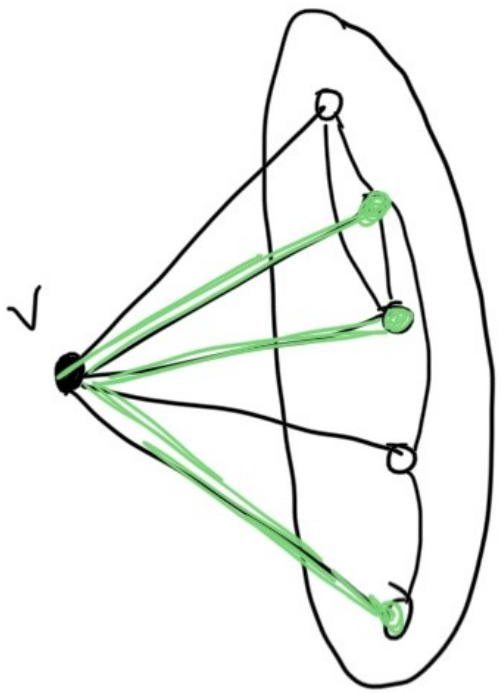

Claim 2

$n_{K_{1,j}}(G) = \sum_{i=1}^{n} \binom{d_G(v_i)}{j}$.

Vertex v appears as the "head" of $K_{1,j}$ exactly $\binom{d_G(v)}{j}$ times where $d_G(v)$ is the degree of v.

Degree sequence: applying the lemma

Suppose G, H have the same ℓ-deck and different degree sequences. From previous slide, $\sum_{i=1}^{n} \binom{d_G(v_i)}{j} = \sum_{i=1}^{n} \binom{d_H(v_i)}{j}$ for $j \in \{0, \ldots, \ell - 1\}$.

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Define $\alpha, \beta \in [n-1]^n$, not related by a permutation, by for $i \in [n]$

$$\alpha_i = d_G(v_i) \text{ and } \beta_i = d_H(v_i).$$

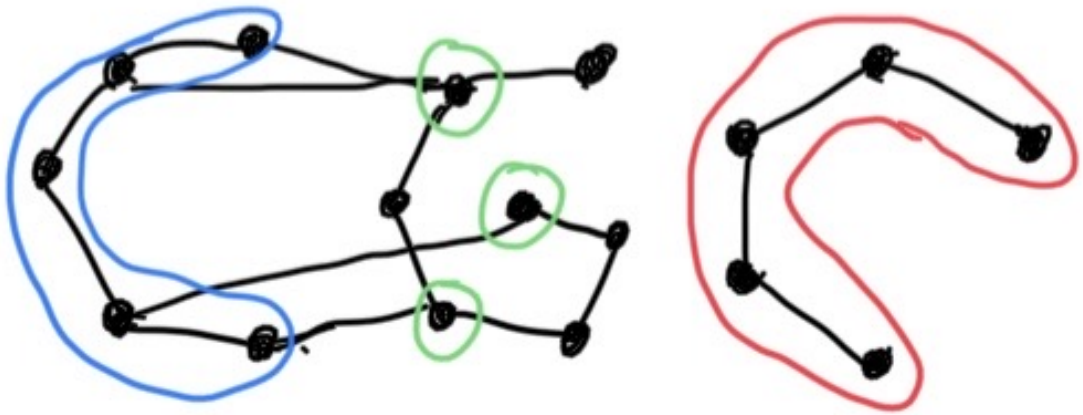
Then we have found

$$\binom{\alpha_1}{j} + \cdots + \binom{\alpha_n}{j} = \binom{\beta_1}{j} + \cdots + \binom{\beta_n}{j} \text{ for all } j \in \{0, \ldots, \ell - 1\}$$

and by the lemma $\ell \leq \sqrt{2(n-1)\log(2n)}$.

Connectedness from small cards (proof sketch)

Suppose we get the ℓ-deck of G for $\ell \geq 0.9n$.



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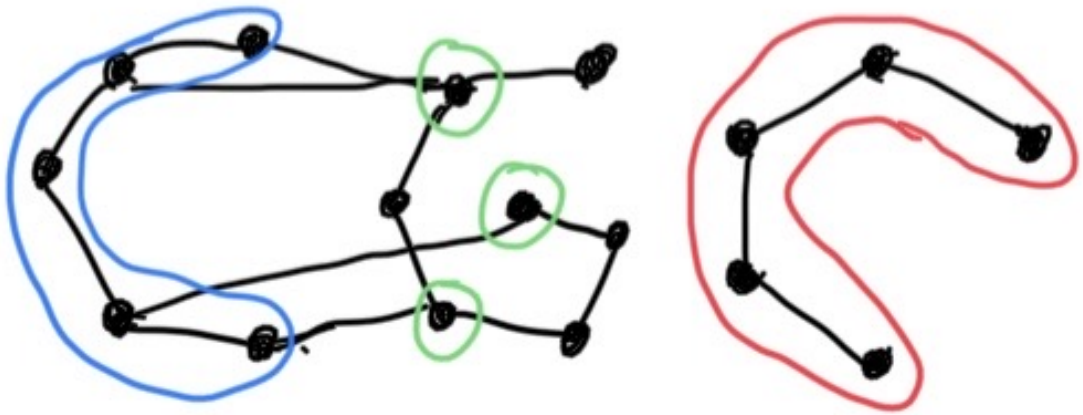

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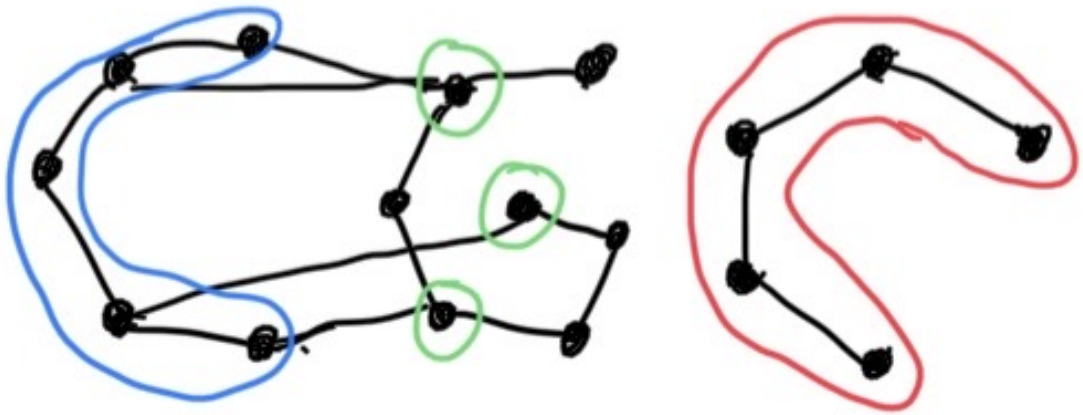

- "Degree" of subgraph = number of adjacent outside vertices. Component = subgraph of degree 0.
- Done if largest component has $\leq \ell - 1$ vertices. (Count number of connected subgraphs on ℓ vertices.)
- For all subgraphs C on $\leq n - (\ell - 1)$ vertices, we reconstruct "degree sequence" to see if there is one of degree 0.

Reconstruction from small cards

Our results

The ℓ-deck of an n-vertex graph G determines the degree sequence if $\ell \geq \sqrt{2n \log(2n)}$ and connectedness if $\ell \geq 0.9n$.

How large does ℓ need to be to determine G?

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How large does ℓ need to be to determine G?

Theorem (Nýdl)

For any $\epsilon > 0$, there are two non-isomorphic graphs on n vertices with the same $\lfloor (1-\epsilon)n \rfloor$-deck.

V. Nýdl, Finite undirected graphs which are not reconstructible from their large cardinality subgraphs. *Discrete Mathematics* **108**:1–3, 1992.

Tree reconstruction from small cards

When can trees be reconstructed from the ℓ-deck?

- Giles[1]: for $\ell \geq n-2$ if $n \geq 5$. (Even the case $\ell = n-1$ is not trivial.)
- Nýdl[2]: conjectured $\ell \geq \lfloor n/2 \rfloor + 1$ for $n \geq 4$. (False for $n = 13$, true for recognition.[3])

[1] W. B. Giles. Reconstructing trees from two point deleted subtrees. *Discrete Mathematics* **15**(4):325–332, 1976.

[2] V. Nýdl. A note on reconstructing of finite trees from small subtrees. *Acta Universitatis Carolinae. Mathematica et Physica* **31**(2):71–74, 1990.

[3] A. V. Kostochka, M. Nahvi, D. B. West, D. Zirlin, Acyclic graphs with at least $2\ell + 1$ vertices are ℓ-recognizable, *arXiv:2103.1215*.

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Theorem (G., Johnston, Scott, Tan)

Trees on n vertices can be reconstructed from the ℓ-deck if $\ell > \frac{8n}{9} + \frac{4}{9}\sqrt{8n+5} + 1$.

---

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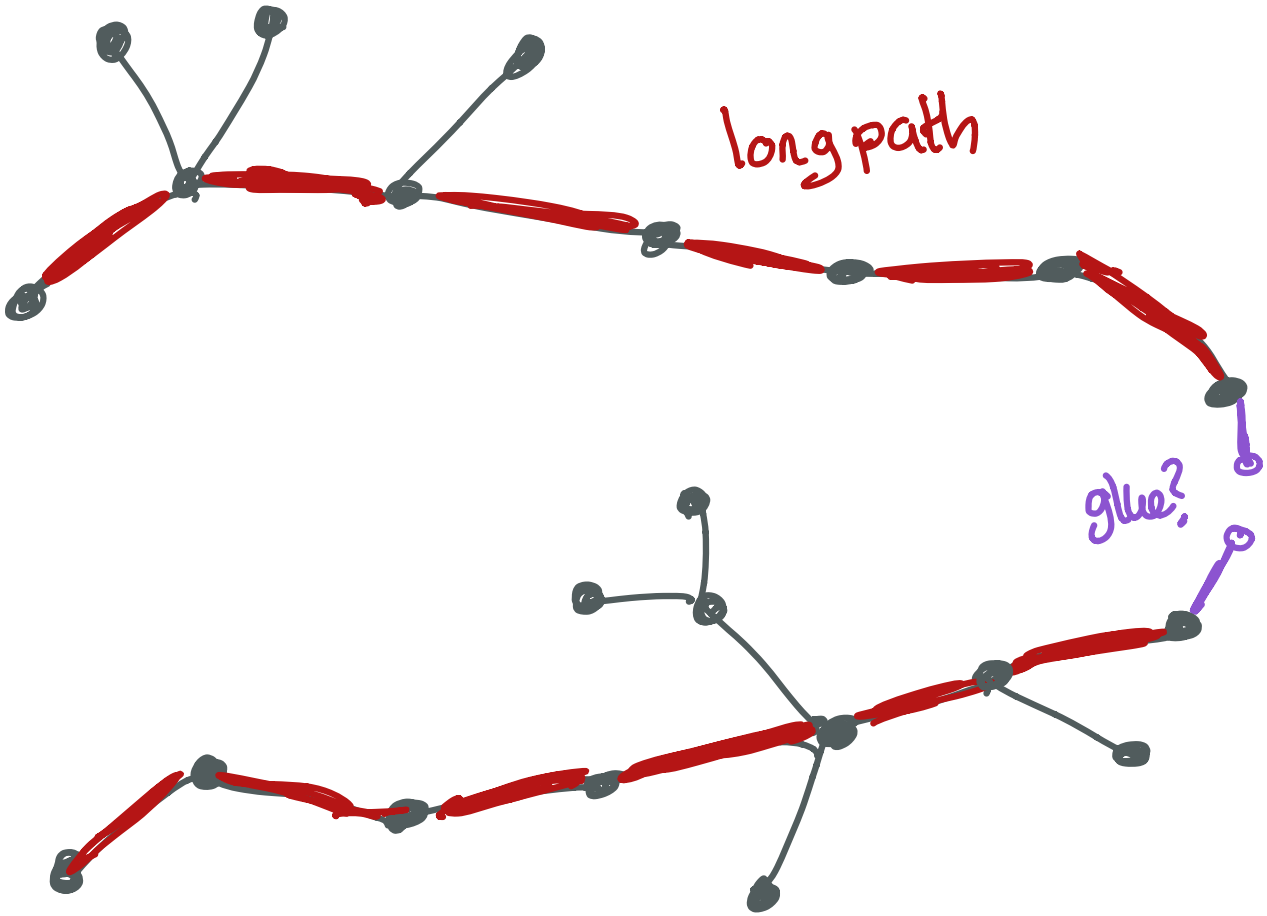
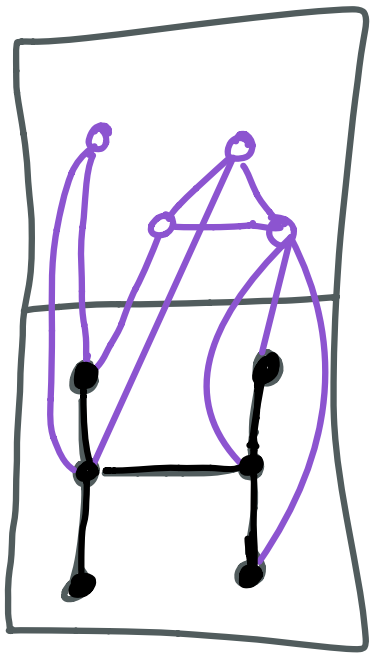
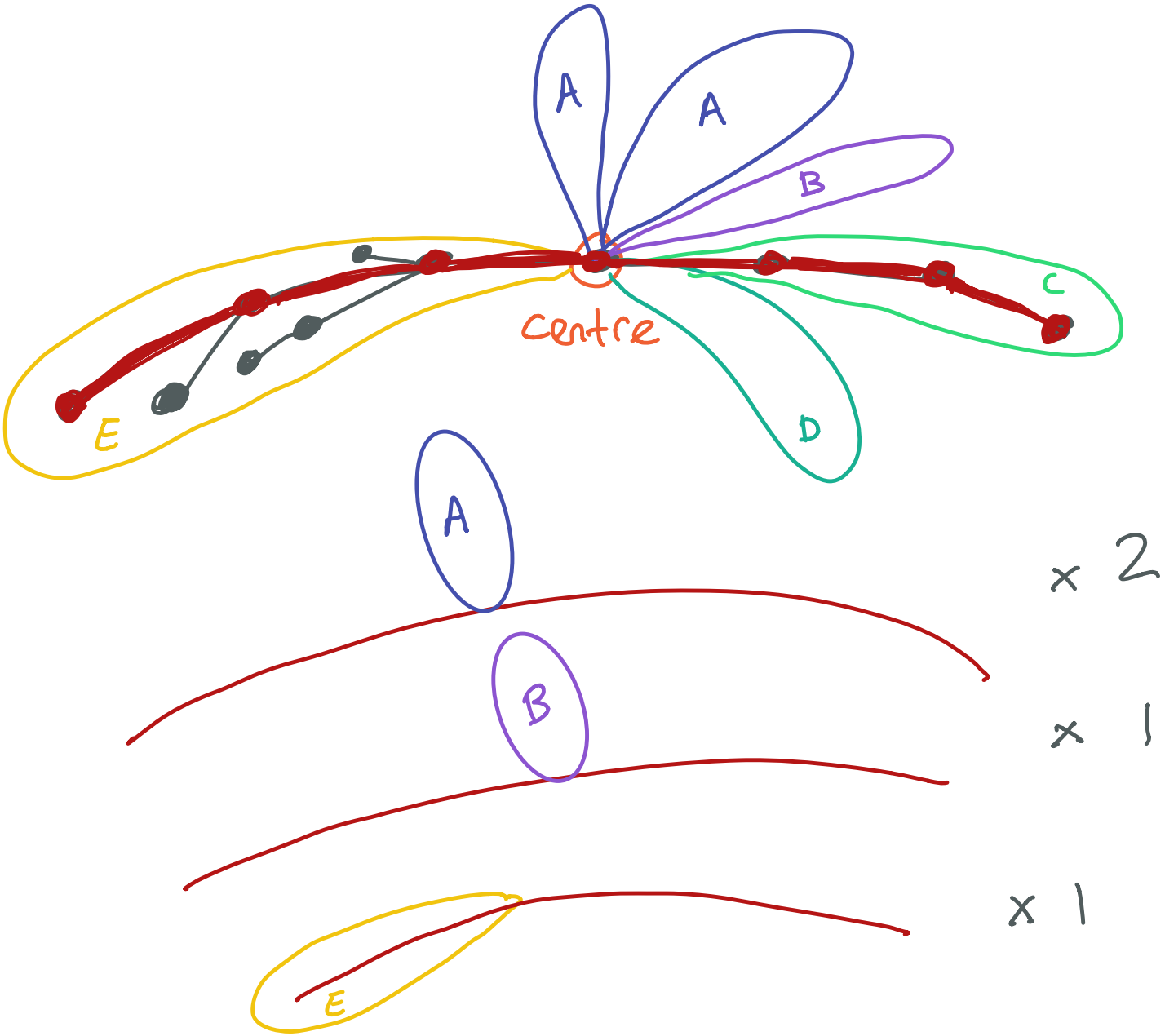
Counting labelled neighbourhoods





Branches as maximal subgraphs



Proof overview tree reconstruction

- If no long path, can find longest path and identify 'centre'. Try to reconstruct 'branches' hanging off the centre.
- Bondy-Hemminger: for certain $\mathcal{F}$, can count the number of maximal $\mathcal{F}$-copies.

- Lemma for counting subgraphs with labelled neighbourhoods.
- If has long path, show can remove an edge e to split into two parts A, B to which lemma applies.

J. Bondy and R. Hemminger. Graph reconstruction – a survey, *Journal of Graph Theory* **1**:227-268, 1977.

Reconstruction from a partial deck

What if less cards are given instead of smaller cards?

- Myrvold[1]: the degree sequence is reconstructible from $n-1$ cards.
- Brown and Fenner[2]: the size of the graph is reconstructible from $n-2$ cards for $n \geq 29$.

[1] W. Myrvold. The degree sequence is reconstructible from $n-1$ cards. *Discr. Math.* **102**:187–196, 1992.

[2] P. Brown and T. Fenner. The number of edges of a graph is reconstructible from any $n-2$ cards. *Discr. Math.* **341**:165–174, 2018.

Our partial deck results

Theorem (G., Guggiari, Scott)

For $k \leq \frac{1}{20}\sqrt{n}$ and n sufficiently large, the number of edges of G can be reconstructed from any $n - k$ cards.

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For $k \leq \frac{1}{20}\sqrt{n}$ and n sufficiently large, the number of edges of G can be reconstructed from any $n - k$ cards.

We can remove a linear number of cards for graphs of bounded degree and planar graphs.

Theorem (G., Johnston, Kupavskii, Meeks, Scott, Tan)

If G has average degree at most d and $n \geq 3$ vertices, then the degree sequence can be reconstructed if $k \leq 10^{-4}d^{-3}n$ cards are missing.

Rounding tricks

Try to reconstruct d_t = number of vertices of degree t.
Let $d_t(G_i)$ = number of vertices of degree t on the ith card.

$$\sum_{i=1}^{n} d_t(G_i) = (n-t-1)d_t + (t+1)d_{t+1}.$$

Important: d_t are integers.

- Can estimate LHS from given cards by $\sum_{i=1}^{n-k} d_t(G_i)$.
- Can 'guess' if d_t, d_{t+1} are small and t is far from 'bad fractions'.
- If know d_{t+1}, can sometimes 'round away' error to compute d_t.

Open problems

- What is the correct asymptotics for reconstructing degree sequence from small cards? We know it is $O(\sqrt{n \log n})$ and $\Omega(\sqrt{\log n})$.
- Close the gaps for recognising connectedness and reconstructing trees.

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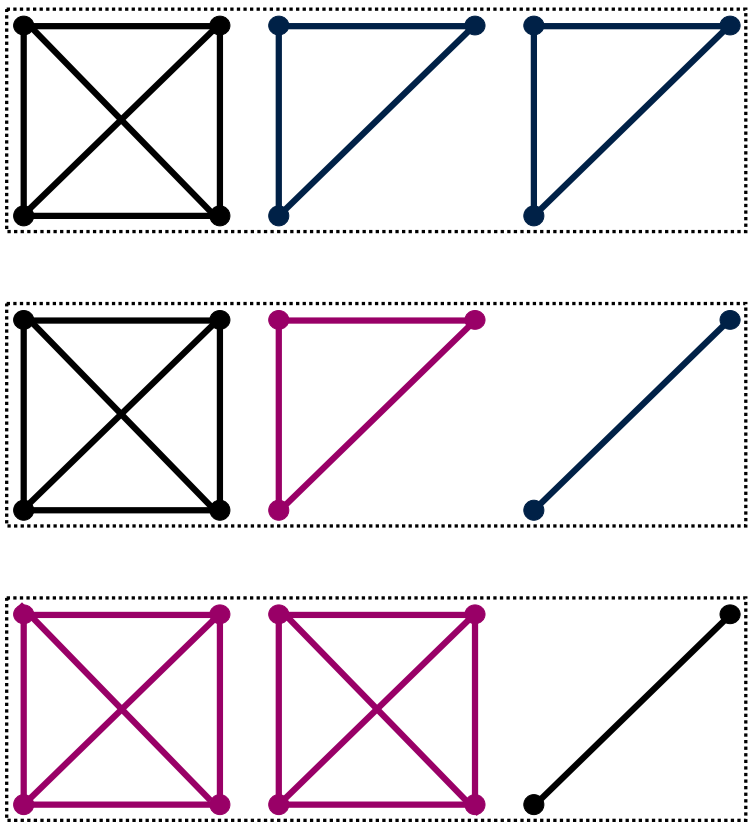
Example with most common cards



The top and bottom graph

$$K_{t+1} \sqcup K_t \sqcup K_t \text{ and } K_{t+1} \sqcup K_{t+1} \sqcup K_{t-1}$$

have $\approx \frac{2}{3}n$ cards in common, all equal to $K_{t+1} \sqcup K_t \sqcup K_{t-1}$ (the middle).

A. Bowler, P. Brown and T. Fenner. Families of pairs of graphs with a large number of common cards. *J. Graph Theory* **63**(2):146-163, 2010.

Open problems

- We can reconstruct the degree sequence and triangle counts even with a linear number of missing cards for planar graphs. Can you do it with a constant number of missing cards for general graphs?

- Are graphs with maximum degree 4 reconstructible?
- (Hard) Are planar graphs reconstructible?

Open problems

- Close the gaps for trees, degree sequence and connectedness.
- Analogue of Nýdl result for partial deck? Are there non-isomorphic graphs with $(1-\epsilon)n$ common cards?
- Can you reconstruct the number of triangles from a partial deck? What about the degree sequence?
- Are planar graphs or graphs with maximum degree 4 reconstructible?

Thank you for listening!

Theorem (Borwein, Erdélyi and Kós)

Any complex polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ has at most $\sqrt{2n \log\left(\frac{|a_0|+|a_1|+\cdots+|a_n|}{\sqrt{|a_0 a_n|}}\right)}$ positive real roots.

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Proof of lemma from theorem: overview

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Consider the polynomial

$$p_{\alpha,\beta}(x) = \sum_{i=1}^{m} x^{\alpha_i} - \sum_{i=1}^{m} x^{\beta_i}.$$

We prove $r + 1 \leq \mathrm{mult}_1(p_{\alpha,\beta}) \leq \sqrt{2n \log(2m)}$.

Proof of lemma from theorem: upper bound

We prove $\mathrm{mult}_1(p_{\alpha,\beta}) = \mathrm{mult}_1\left(\sum_{j=0}^{n'} a_j x^j\right) \leq \sqrt{2n\log(2m)}$, where

$$p_{\alpha,\beta}(x) = \sum_{i=1}^{m} x^{\alpha_i} - \sum_{i=1}^{m} x^{\beta_i} = x^k \sum_{j=0}^{n'} a_j x^j$$

with $0 \leq n' \leq n$ and $a_0, a_{n'} \neq 0$, where

$$a_j = |\{i : \alpha_i = j + k\}| - |\{i : \beta_i = j + k\}|.$$

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By the theorem, the number of positive real roots of $\sum_{j=1}^{n'} a_j x^j$ is at most

$$\sqrt{2n' \log\left(\frac{|a_0| + |a_1| + \cdots + |a_{n'}|}{\sqrt{|a_0 a_{n'}|}}\right)}.$$

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For $j \in \{0, \ldots, r\}$, we find

$$\left| \left(\frac{d}{dx^j} \left[\sum_{i=1}^m x^{\alpha_i} - \sum_{i=1}^m x^{\beta_i} \right] \right) \right|_{x=1} = \sum_{i=1}^m j! \binom{\alpha_i}{j} - \sum_{i=1}^m j! \binom{\beta_i}{j} = 0.$$

Bounded average degree ideas

- Suppose v_1 has small degree and $G_1 = G - v_1$. Then $d_{G_1}(v) \approx d_G(v)$.
- Proof 1: reconstruct $n_t(v) =$ number of neighbours of degree t; we can reconstruct $n_t(v) - n_t(w)$ from $G - v$ and $G - w$.
- Proof 2: estimate $\sum_v d_t(G - v)$ by

$$\sum_{v : d_{G_1}(v) > 100d^2} d_t(G_1 - v) + \sum_{v \in \mathcal{C} : d_G(v) \leq 100d^2} d_t(G - v) + |\text{others}| d_t(G_1).$$

Computation for connectedness

Let $C_1, \ldots, C_m$ be copies of C in G, with α_i the degree of C_i. We may recontruct $\sum_{i=1}^{m} \binom{\alpha_i}{j}$ for $j \in \{0, \ldots, r\}$ where $r = \ell - |V(C)|$. Borwein-Ingalls Lemma implies reconstruction is possible if $r \geq \sqrt{2n \log(2m)}$. Suppose $|V(C)| = \epsilon n \geq 1$. Then

$$m \leq \binom{n}{|V(C)|} \leq \left(\frac{en}{\epsilon n}\right)^{\epsilon n}$$

and for ϵ small we find

$$\begin{aligned}
\sqrt{2n \log(2m)} &\leq \sqrt{2n \log\left(2\left(\frac{en}{\epsilon n}\right)^{\epsilon n}\right)} \\
&\leq n\sqrt{2\epsilon \log\left(2e/\epsilon\right)} \\
&\leq \ell - |V(C)| = r.
\end{aligned}$$

Most important equation

Recall $G_i := G - v_i$ and $G_1, \ldots, G_{n-k}$ have been given.

$$\sum_{i=1}^{n} d_t(G_i) = (n-1-t)d_t + (t+1)d_{t+1}. \tag{1}$$

Vertex of degree t on card G_i can be

- vertex v of degree t (whenever v not connected to v_i)
- vertex u of degree $t+1$ (whenever u connected to v_i)

Since $d_t(G_i) \leq d_t + d_{t+1}$ for all i, can calculate (1) up to

$$\underbrace{\sum_{i=n-k+1}^{n} d_t(G_i)}_{\text{contribution of missing cards}} \leq \underbrace{k}_{\text{number of missing cards}} (d_t + d_{t+1}).$$

Estimating d_{t-1} from d_t

Recall

$$\sum_{i=1}^{n} d_t(G_i) = (n-1-t)d_t + (t+1)d_{t+1}.$$

Hence if d_{t+1} is known, we can calculate

$$d'_t = \frac{1}{n-1-t}\left(\sum_{i=1}^{n-k} d_t(G_i) - (t+1)d_{t+1}\right).$$

By rounding, we can get rid of the error $0 \leq d_t - d'_t \leq \frac{k(d_t+d_{t+1})}{n-1-t}$ if, say, $d_t + d_{t+1} \leq 4\sqrt{n}$ and $t \leq \frac{2}{3}n$.

Computing d_t exactly for most t

We can rewrite our magic equation

$$\sum_{i=1}^{n} d_t(G_i) = (n-1-t)d_t + (t+1)d_{t+1}$$

for $q = \frac{t+1}{n}$ to

$$\underbrace{\frac{1}{n}\sum_{i=1}^{n-k} d_t(G_i)}_{\text{known}} = (1-q)d_t + qd_{t+1} - \underbrace{\frac{1}{n}\sum_{i=n-k+1}^{n} d_t(G_i)}_{\text{small error}}.$$

Idea: For most q (i.e. t), the values $(1-q)a + qb$ are far enough apart to give unique "small" option for (a, b).
Moreover, for most t, $d_t + d_{t+1}$ is "small".

Proof overview

- Approximation $\widetilde{m}$ of m, the number of edges of G
- Approximation $\widetilde{d}_t$ of d_t, the number of vertices of degree t in G, by computing

$$\widetilde{d(v_i)} = \widetilde{m} - e(G_i)$$

  Two errors: "shift" $\alpha = \widetilde{m} - m$ and "missing values"
- Second approximation d'_t of d_t.
  - If know d_t and it is small, then can either compute d_{t-1} (or d_{t+1}) exactly or discover it is large.
  - Can calculate d_t exactly for all but $o(n)$ of the t.
- Goal: If d_t large, and the known region next to it contains many consecutive small values $d_{t+a}, \ldots, d_{t+b}$, then these can be used to find the "shift" in $\widetilde{d}_t$.

Discovering the shift

Complementarity principle: cards of the complement $\overline{G}$ are $\overline{G_1}, \ldots, \overline{G_n}$.
Moreover, we have that $d_t(G) = d_{n-1-t}(\overline{G})$ for any $t \in \{0, \ldots, n-1\}$.

We "guess" many d_t exactly.
In the middle ($n/3 \leq t \leq 2n/3$) this allows us to compute d_{t+1} and d_{t-1} either exactly, or identify as large.