Chromatic zeros of series-parallel graphs

Jeroen Huijben

joint work with Ferenc Bencs and Guus Regts

arXiv:2204.10038

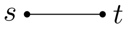
University of Amsterdam
Korteweg de Vries Institute for Mathematics

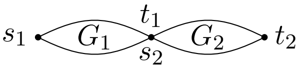
May 2022

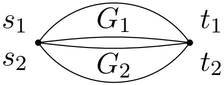
Outline

Outline

Series-parallel graphs

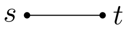
- 

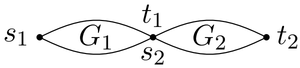
is a series-parallel graph
- 

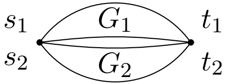
is a series-parallel graph $G_1 \bowtie G_2$
- 

is a series-parallel graph $G_1 \parallel G_2$

Series-parallel graphs

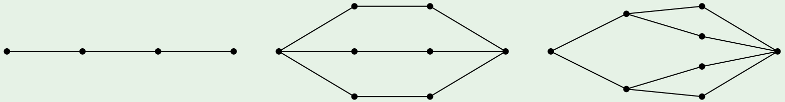
- 

is a series-parallel graph
- 

is a series-parallel graph $G_1 \bowtie G_2$
- 

is a series-parallel graph $G_1 \parallel G_2$

Example



Chromatic zeros

Chromatic polynomial

$$Z(G;q) = \#\text{ proper } q\text{-colourings of } G$$
$$= \sum_{F\subseteq E(G)} (-1)^{|F|} q^{k(F)}$$

Chromatic zeros

Chromatic polynomial

$$\begin{aligned} Z(G;q) &= \#\text{ proper } q\text{-colourings of } G \\ &= \sum_{F\subseteq E(G)} (-1)^{|F|} q^{k(F)} \end{aligned}$$

(Sokal, 2004) Chromatic zeros are dense in the complex plane.
Chromatic zeros of generalized theta graphs are dense outside $|q-1| \leq 1$.

Chromatic zeros

Chromatic polynomial

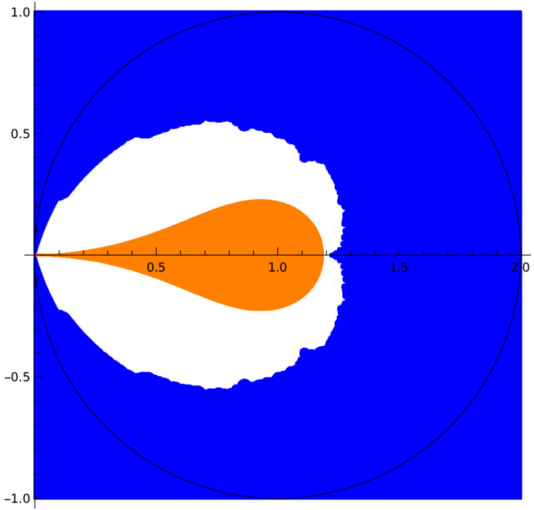
$$Z(G;q) = \#\text{ proper } q\text{-colourings of } G$$
$$= \sum_{F\subseteq E(G)} (-1)^{|F|} q^{k(F)}$$

(Sokal, 2004) Chromatic zeros are dense in the complex plane.
Chromatic zeros of generalized theta graphs are dense outside $|q-1| \leq 1$.

Question

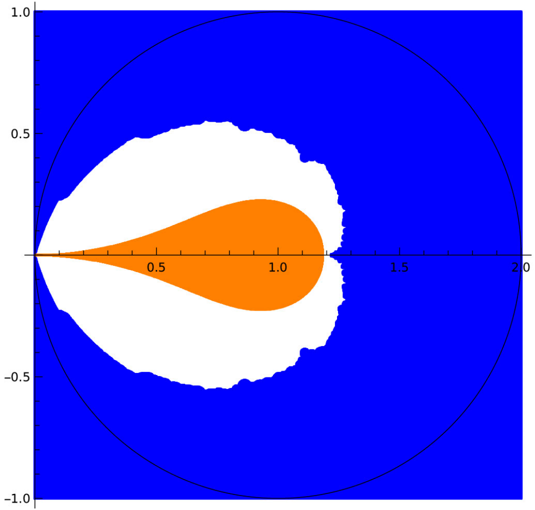
Where are the chromatic zeros of all series-parallel graphs?

Chromatic zeros



Chromatic zeros

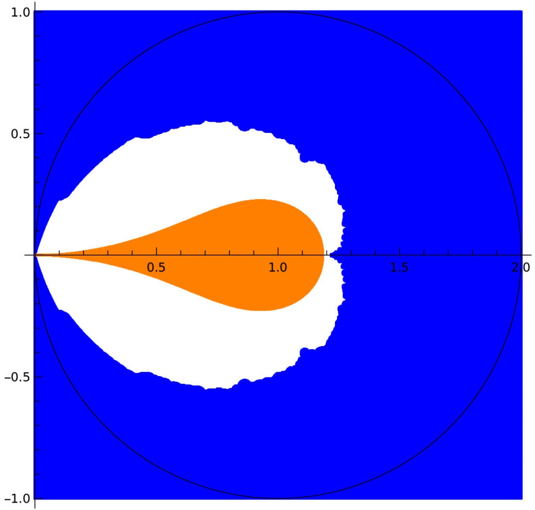
Blue = Chromatic zeros
(We prove: the region $\Re(q) > 3/2$ is blue.)



Chromatic zeros

Blue = Chromatic zeros
(We prove: the region $\Re(q) > 3/2$ is blue.)

Orange = Zero-free
(We prove: there exists a punctured open around $(0, 32/27)$ which is zero-free.)



Chromatic zeros

Conjecture (Sokal, 2005)

Consider all graphs with vertex degree $\leq \Delta$, except possibly one vertex. The real part of the chromatic zeros of these graphs is $\leq \Delta$.

Chromatic zeros

Conjecture (Sokal, 2005)

Consider all graphs with vertex degree $\leq \Delta$, except possibly one vertex. The real part of the chromatic zeros of these graphs is $\leq \Delta$.

For Δ large enough, this is false!

Chromatic zeros

Conjecture (Sokal, 2005)

Consider all graphs with vertex degree $\leq \Delta$, except possibly one vertex. The real part of the chromatic zeros of these graphs is $\leq \Delta$.

For Δ large enough, this is false!
We use a connection between the chromatic polynomial of leaf joined trees, and the independence polynomial of trees.

Outline

Effective interaction

We split

$$Z(G;q) = Z^{\mathrm{same}}(G;q) + Z^{\mathrm{dif}}(G;q)$$

and define the *effective interaction*

$$y_G(q) = (q-1)\frac{Z^{\mathrm{same}}(G;q)}{Z^{\mathrm{dif}}(G;q)}.$$

Effective interaction

We split

$$Z(G;q) = Z^{\text{same}}(G;q) + Z^{\text{dif}}(G;q)$$

and define the *effective interaction*

$$y_G(q) = (q-1)\frac{Z^{\text{same}}(G;q)}{Z^{\text{dif}}(G;q)}.$$

With Möbius transformation $f_q(y) = 1 + \frac{q}{y-1}$, we have

$$y_{G_1 \| G_2} = y_{G_1} \cdot y_{G_2}, \qquad f_q(y_{G_1 \bowtie G_2}) = f_q(y_{G_1}) \cdot f_q(y_{G_2}), \qquad f_q(f_q(y)) = y.$$

The effective interaction detects chromatic zeros

Lemma

For $q \in \mathbb{C} \setminus \{0,1,2\}$ *the following are equivalent*

- $Z(G;q) = 0$ *for some series-parallel graph* G*;*
- $y_G \in \{0, 1-q, \infty\}$ *for some series-parallel* graph* G*;*
- $y_G = 1-q$ *for some series-parallel* graph* G*.*

The effective interaction detects chromatic zeros

Lemma

For $q \in \mathbb{C} \setminus \{0, 1, 2\}$ *the following are equivalent*

- $Z(G; q) = 0$ *for some series-parallel graph* G*;*
- $y_G \in \{0, 1 - q, \infty\}$ *for some series-parallel* graph* G*;*
- $y_G = 1 - q$ *for some series-parallel* graph* G*.*

To find chromatic zeros, we use Montel's theorem

Theorem (Montel)

Let $\mathcal{F}$ *be a family of holomorphic functions* $f : U \to \hat{\mathbb{C}}$ *that avoids three distinct values* $a, b, c \in \hat{\mathbb{C}}$*. Then* $\mathcal{F}$ *is a normal family.*

(A family of functions $\mathcal{F}$ is *normal* if every sequence in $\mathcal{F}$ has a subsequence which converges on compact sets.)

Non-normal families

Theorem

If for $q_0 \in \mathbb{C} \setminus \{0, 1, 2\}$ *there exists a series-parallel graph* G_0 *with* $|y_{G_0}(q_0)| > 1$ *or* $|f_{q_0}(y_{G_0}(q_0))| > 1$, *then the family* $\{q \mapsto y_G(q)\}_{G \text{ series-parallel}}$ *is non-normal near* q_0.

Non-normal families

Theorem

If for $q_0 \in \mathbb{C} \setminus \{0, 1, 2\}$ *there exists a series-parallel graph* G_0 *with* $|y_{G_0}(q_0)| > 1$ *or* $|f_{q_0}(y_{G_0}(q_0))| > 1$, *then the family* $\{q \mapsto y_G(q)\}_{G \text{ series-parallel}}$ *is non-normal near* q_0.

Proof sketch, second case.

For q in a small nbhd around q_0 and $N \to \infty$, $f_q(y)^N$ approaches ∞ from all direction.

Non-normal families

Theorem

If for $q_0 \in \mathbb{C} \setminus \{0,1,2\}$ *there exists a series-parallel graph* G_0 *with* $|y_{G_0}(q_0)| > 1$ *or* $|f_{q_0}(y_{G_0}(q_0))| > 1$, *then the family* $\{q \mapsto y_G(q)\}_{G \text{ series-parallel}}$ *is non-normal near* q_0.

Proof sketch, second case.

For q in a small nbhd around q_0 and $N \to \infty$, $f_q(y)^N$ approaches ∞ from all direction. Then $f_q(f_q(y)^N)$ approaches 1 from all directions.

Non-normal families

Theorem

If for $q_0 \in \mathbb{C} \setminus \{0,1,2\}$ there exists a series-parallel graph G_0 with $|y_{G_0}(q_0)| > 1$ or $|f_{q_0}(y_{G_0}(q_0))| > 1$, then the family $\{q \mapsto y_G(q)\}_{G \text{ series-parallel}}$ is non-normal near q_0.

Proof sketch, second case.

For q in a small nbhd around q_0 and $N \to \infty$, $f_q(y)^N$ approaches ∞ from all direction.
Then $f_q(f_q(y)^N)$ approaches 1 from all directions.
For N large enough and $m \to \infty$, the powers $f_q(f_q(y)^N)^m$ converge to 0 inside the unit disk, and to ∞ outside the unit disk. Therefore this family is non-normal. □

Non-normal families

Theorem

If for $q_0 \in \mathbb{C} \setminus \{0,1,2\}$ *there exists a series-parallel graph* G_0 *with* $|y_{G_0}(q_0)| > 1$ *or* $|f_{q_0}(y_{G_0}(q_0))| > 1$, *then the family* $\{q \mapsto y_G(q)\}_{G \text{ series-parallel}}$ *is non-normal near* q_0.

Proof sketch, second case.

For q in a small nbhd around q_0 and $N \to \infty$, $f_q(y)^N$ approaches ∞ from all direction.
Then $f_q(f_q(y)^N)$ approaches 1 from all directions.
For N large enough and $m \to \infty$, the powers $f_q(f_q(y)^N)^m$ converge to 0 inside the unit disk, and to ∞ outside the unit disk. Therefore this family is non-normal. □

This recovers Sokal's result, because $f_q(0) = 1 - q$.

Non-normal families

Theorem

If for $q_0 \in \mathbb{C} \setminus \{0,1,2\}$ *there exists a series-parallel graph* G_0 *with* $|y_{G_0}(q_0)| > 1$ *or* $|f_{q_0}(y_{G_0}(q_0))| > 1$, *then the family* $\{q \mapsto y_G(q)\}_{G \text{ series-parallel}}$ *is non-normal near* q_0.

Proof sketch, second case.

For q in a small nbhd around q_0 and $N \to \infty$, $f_q(y)^N$ approaches ∞ from all direction.
Then $f_q(f_q(y)^N)$ approaches 1 from all directions.
For N large enough and $m \to \infty$, the powers $f_q(f_q(y)^N)^m$ converge to 0 inside the unit disk, and to ∞ outside the unit disk. Therefore this family is non-normal. □

This recovers Sokal's result, because $f_q(0) = 1 - q$.
Yields zeros for $\Re(q) > \frac{3}{2}$, because $f_q(f_q(0)^2) = \frac{q-1}{q-2}$.

Outline

Construction of leaf joined trees

We will consider the following family $\mathcal{T}_d$ of *leaf-joined trees*

- $K_2 =$  is in $\mathcal{T}_d$.
- If $T_1, \ldots, T_d$ are in $\mathcal{T}_d$, then $(K_2 \bowtie T_1) \parallel (K_2 \bowtie T_2) \parallel \cdots \parallel (K_2 \bowtie T_d)$ is in $\mathcal{T}_d$.

Construction of leaf joined trees

We will consider the following family $\mathcal{T}_d$ of *leaf-joined trees*

- $K_2 =$ •——• is in $\mathcal{T}_d$.
- If $T_1, \dots, T_d$ are in $\mathcal{T}_d$, then $(K_2 \bowtie T_1) \parallel (K_2 \bowtie T_2) \parallel \cdots \parallel (K_2 \bowtie T_d)$ is in $\mathcal{T}_d$.

Note that, except in one vertex, the degree in every leaf joined tree in $\mathcal{T}_d$ is at most $d + 1$.

Construction of leaf joined trees

We will consider the following family $\mathcal{T}_d$ of *leaf-joined trees*

- $K_2 =$  is in $\mathcal{T}_d$.
- If $T_1, \ldots, T_d$ are in $\mathcal{T}_d$, then $(K_2 \bowtie T_1) \parallel (K_2 \bowtie T_2) \parallel \cdots \parallel (K_2 \bowtie T_d)$ is in $\mathcal{T}_d$.

Note that, except in one vertex, the degree in every leaf joined tree in $\mathcal{T}_d$ is at most $d+1$.

We now look at the effective interaction of this iteration

$$y = \prod_{i=1}^{d} f_q(f_q(0) f_q(y_i)),$$

Construction of leaf joined trees

We will consider the following family $\mathcal{T}_d$ of *leaf-joined trees*

- $K_2 =$  is in $\mathcal{T}_d$.
- If $T_1, \dots, T_d$ are in $\mathcal{T}_d$, then $(K_2 \bowtie T_1) \parallel (K_2 \bowtie T_2) \parallel \cdots \parallel (K_2 \bowtie T_d)$ is in $\mathcal{T}_d$.

Note that, except in one vertex, the degree in every leaf joined tree in $\mathcal{T}_d$ is at most $d+1$.

We now look at the effective interaction of this iteration

$$y = \prod_{i=1}^{d} f_q(f_q(0) f_q(y_i)),$$

$$\frac{y}{q-2} = \frac{\lambda(q,d)}{\prod_{i=1}^{d}(1 + \frac{y_i}{q-2})}, \qquad \text{where } \lambda(q,d) := \frac{(q-1)^d}{(q-2)^{d+1}}.$$

Independence ratios

Consider the following family $\mathcal{R}_d$ of *independence ratios*

- $\lambda \mapsto 0$ is in $\mathcal{R}_d$;
- if $r_1, \dots, r_d \in \mathcal{R}_d$, then $\lambda \mapsto \frac{\lambda}{\prod_{i=1}^{d}(1+r_i(\lambda))}$ is in $\mathcal{R}_d$.

Independence ratios

Consider the following family $\mathcal{R}_d$ of *independence ratios*

- $\lambda \mapsto 0$ is in $\mathcal{R}_d$;
- if $r_1, \dots, r_d \in \mathcal{R}_d$, then $\lambda \mapsto \frac{\lambda}{\prod_{i=1}^{d}(1+r_i(\lambda))}$ is in $\mathcal{R}_d$.

In the context of the independence polynomial, normality of this family has been studied carefully in the last years.

Independence ratios

Consider the following family $\mathcal{R}_d$ of *independence ratios*

- $\lambda \mapsto 0$ is in $\mathcal{R}_d$;
- if $r_1, \ldots, r_d \in \mathcal{R}_d$, then $\lambda \mapsto \frac{\lambda}{\prod_{i=1}^{d}(1+r_i(\lambda))}$ is in $\mathcal{R}_d$.

In the context of the independence polynomial, normality of this family has been studied carefully in the last years.
If $q \mapsto \lambda(q, d)$ is locally injective at q_0, and the family $\mathcal{R}_d$ is non-normal near $\lambda(q_0, d)$, then we find chromatic zeros near q_0.

Independence ratios

Consider the following family $\mathcal{R}_d$ of *independence ratios*

- $\lambda \mapsto 0$ is in $\mathcal{R}_d$;
- if $r_1, \ldots, r_d \in \mathcal{R}_d$, then $\lambda \mapsto \frac{\lambda}{\prod_{i=1}^{d}(1+r_i(\lambda))}$ is in $\mathcal{R}_d$.

In the context of the independence polynomial, normality of this family has been studied carefully in the last years.
If $q \mapsto \lambda(q, d)$ is locally injective at q_0, and the family $\mathcal{R}_d$ is non-normal near $\lambda(q_0, d)$, then we find chromatic zeros near q_0.

Theorem (Bencs, Buys, Peters, 2021)

For d large enough, there exists $q_0 > d + 1$ such that $\mathcal{R}_d$ is non-normal near $\lambda(q_0, d)$.

Further questions

We proved: interactions escape unit disk $\implies$ non-normal family of interactions $\implies$ chromatic zeros.

Further questions

We proved: interactions escape unit disk $\implies$ non-normal family of interactions $\implies$ chromatic zeros.
Are any of the reverse implications true? Will this give a complete picture of the chromatic roots?

Further questions

We proved: interactions escape unit disk $\implies$ non-normal family of interactions $\implies$ chromatic zeros.

Are any of the reverse implications true? Will this give a complete picture of the chromatic roots?

Are there larger families of graphs for which we can find zero-free regions?

Further questions

We proved: interactions escape unit disk $\implies$ non-normal family of interactions $\implies$ chromatic zeros.
Are any of the reverse implications true? Will this give a complete picture of the chromatic roots?
Are there larger families of graphs for which we can find zero-free regions?

Can we settle Sokal's conjecture for all Δ?

Thank you for your attention!