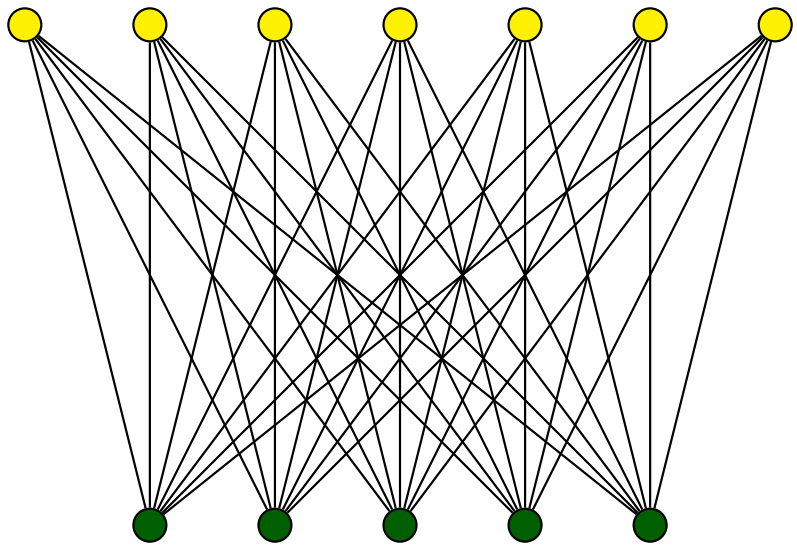
New lower bounds on crossing numbers of $K_{m,n}$ from permutation modules and semidefinite programming

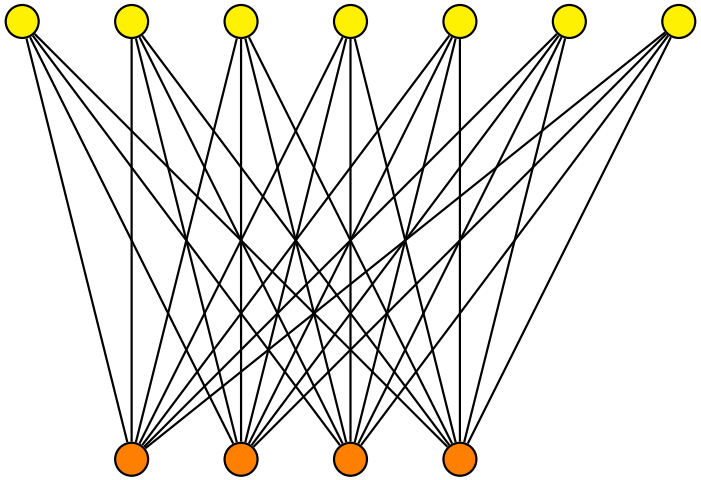
Sven Polak

CWI



Joint work with Daniel Brosch (Tilburg University)

Crossing numbers of $K_{m,n}$



The graph $K_{7,4}$

- Minimum number of crossings? (Turán, 1940s)

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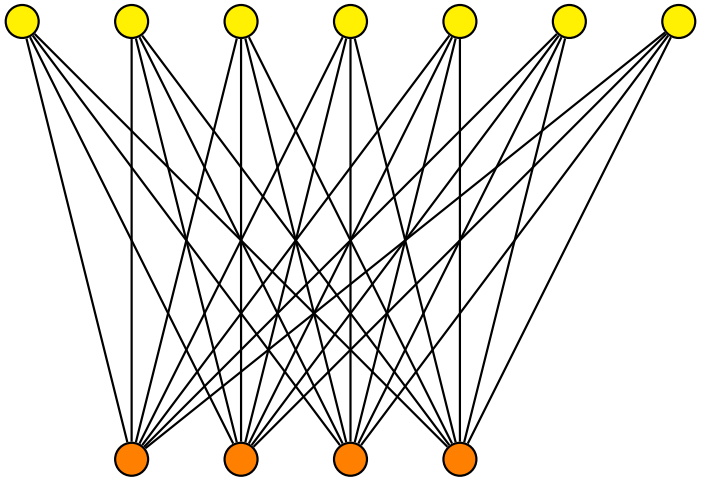

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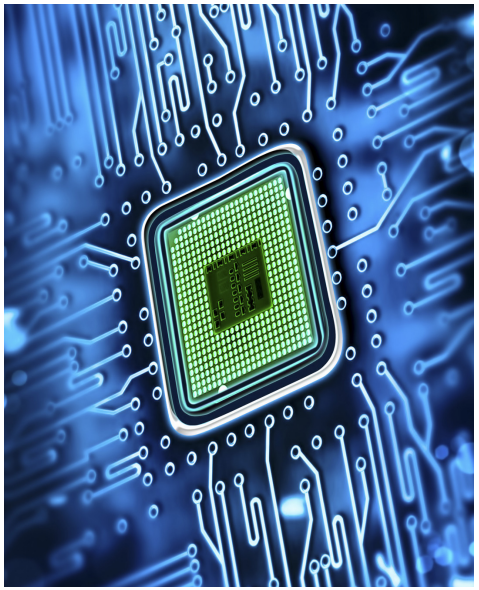

Computer chip

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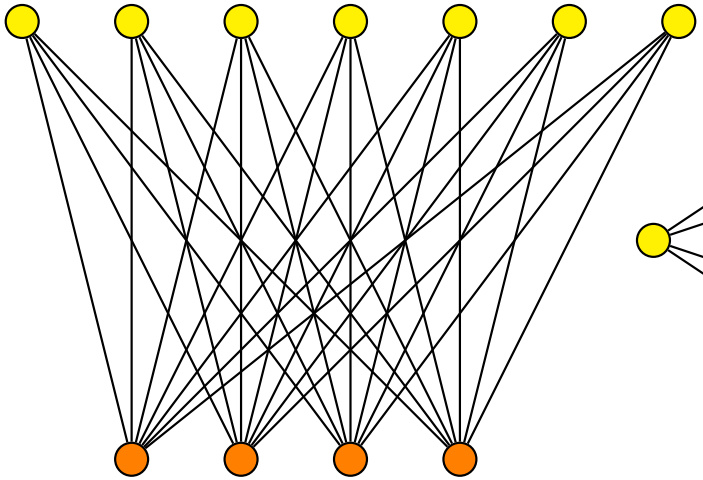

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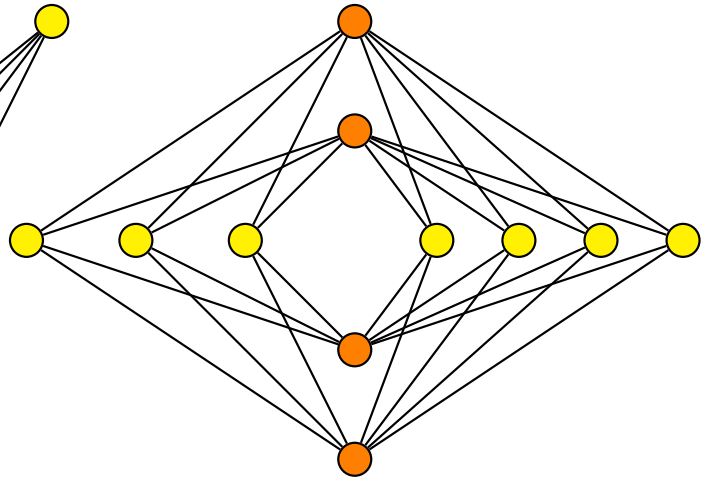
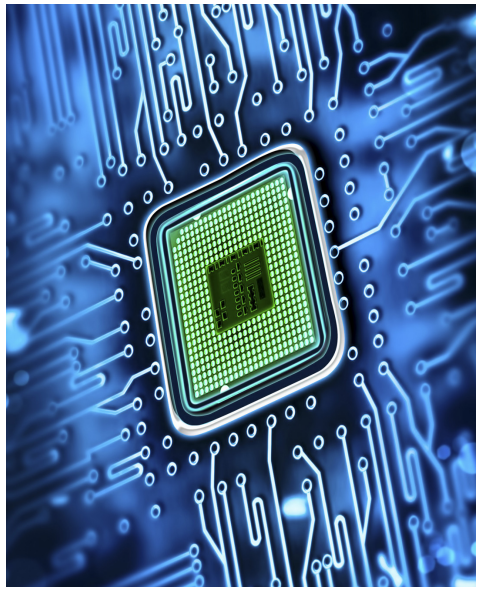


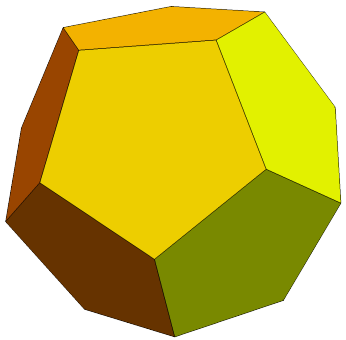
Optimal drawing



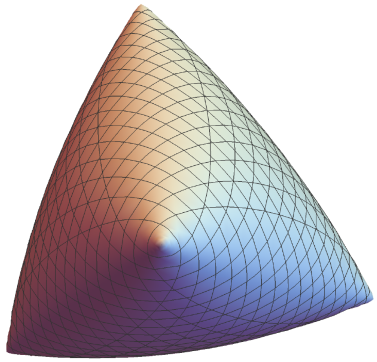
Computer chip

- Minimum number of crossings? (Turán, 1940s)
- $\mathrm{cr}(K_{m,n}) = \lfloor \frac{(m-1)^2}{4} \rfloor \lfloor \frac{(n-1)^2}{4} \rfloor$? (Zarankiewicz's conjecture, 1956)

SDP: strong approximations for hard problems



Linear optimization
Classical (1940s)
Efficient in theory/practice



Semidefinite optimization (**SDP**)
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László Lovász
Abel prize 2021

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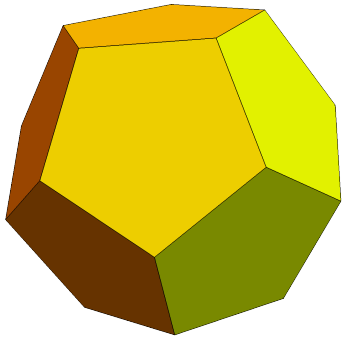

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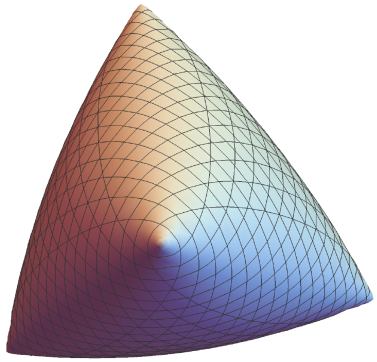

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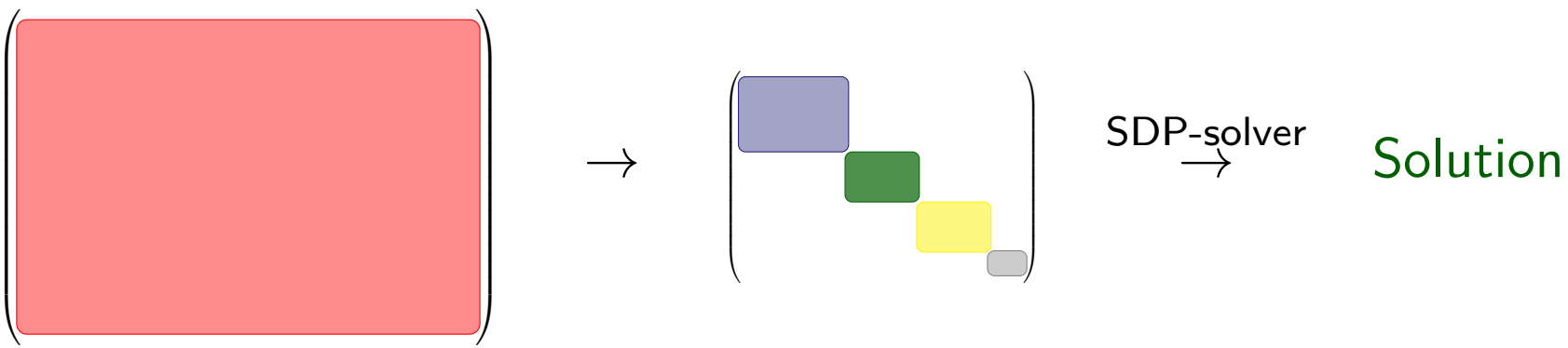
Problem

In practice, often hard to solve SDPs. **Matrix size** is limiting factor.



Cyclic orders

Let $\{1, \ldots, m\}$ and $\{b_1, \ldots, b_n\}$ be the sides of the bipartition of $K_{m,n}$. Given a drawing of $K_{m,n}$, define for each vertex b_i:

$$\gamma(b_i) := \text{ the cyclic order in which edges from } b_i \text{ go to } \{1, \ldots, m\}$$

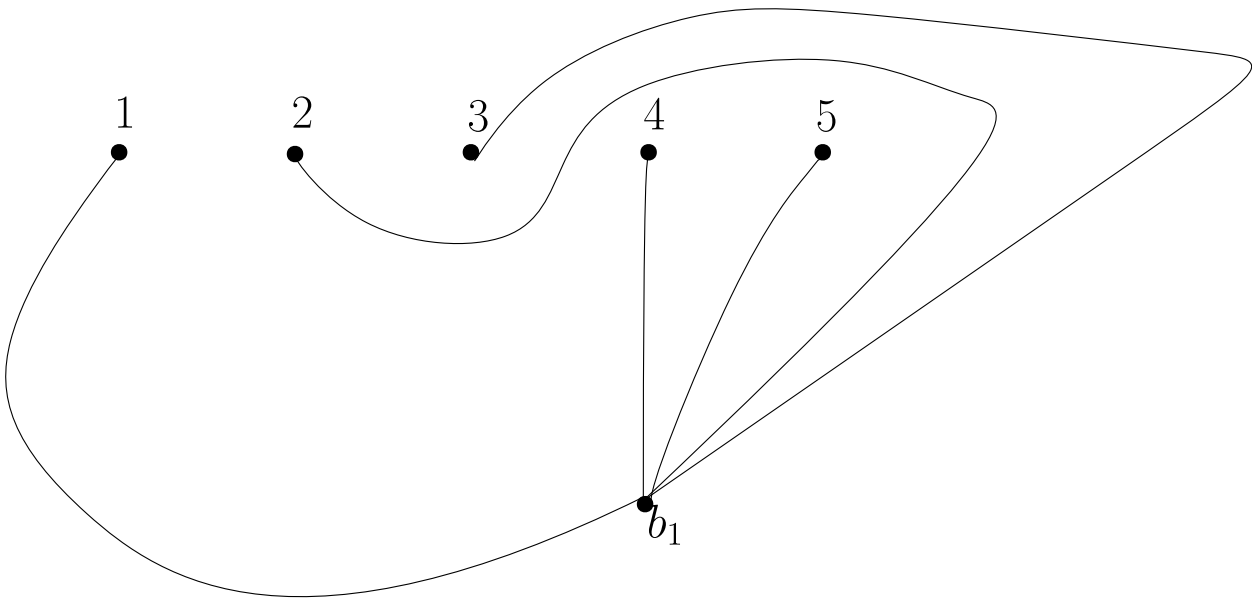


Figure: $\gamma(b_1) = (14523)$.

Cyclic orders

Lemma (Kleitman; 1970, Woodall; 1993)

#crossings of edges incident with b_i and $b_j \geq$
min. #swaps of adjacent elements of $\gamma(b_i)$ to change $\gamma(b_i)$ into $\gamma(b_j)^{-1}$.

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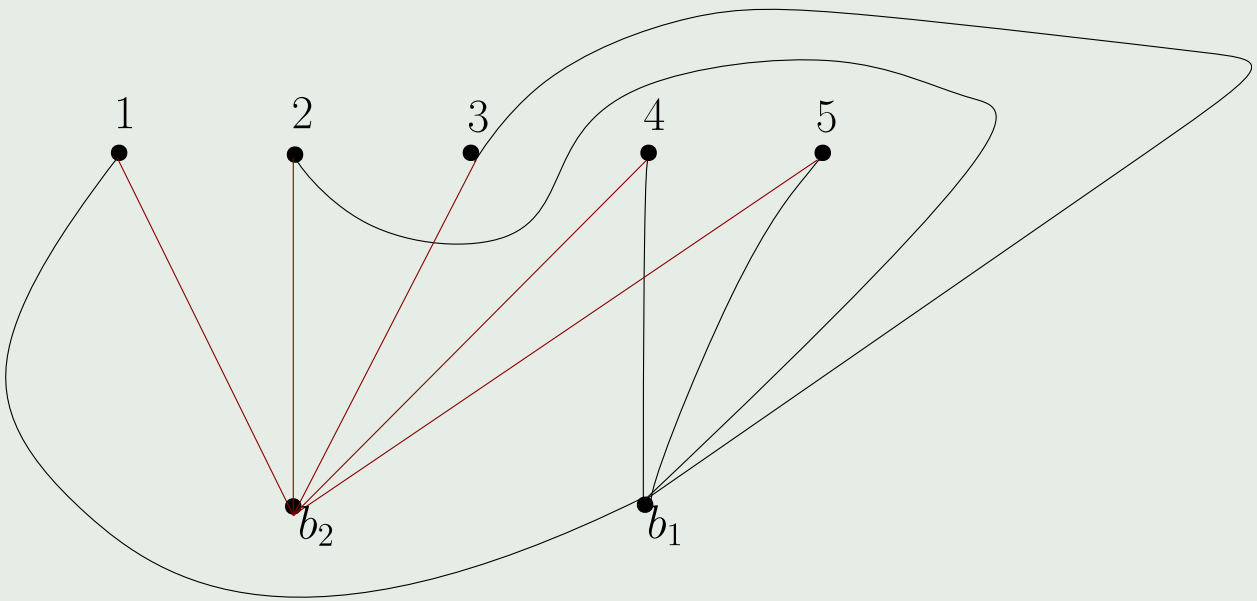

Figure: $\gamma(b_2) = (12345)$ and $\gamma(b_1)^{-1} = (14523)^{-1} = (13254)$.
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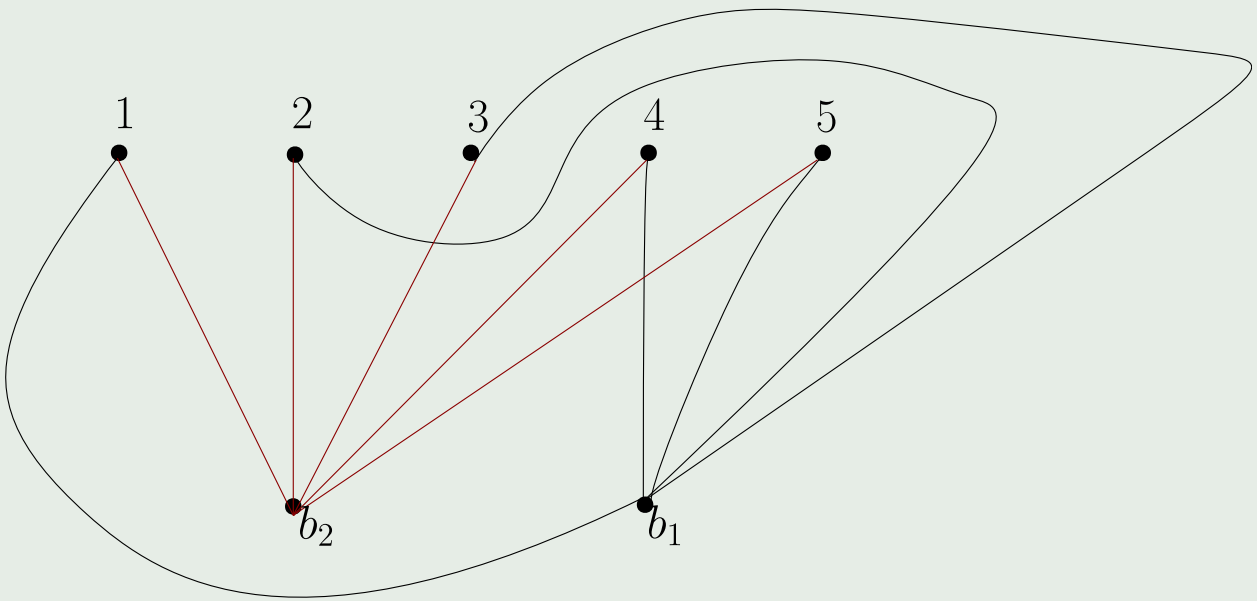

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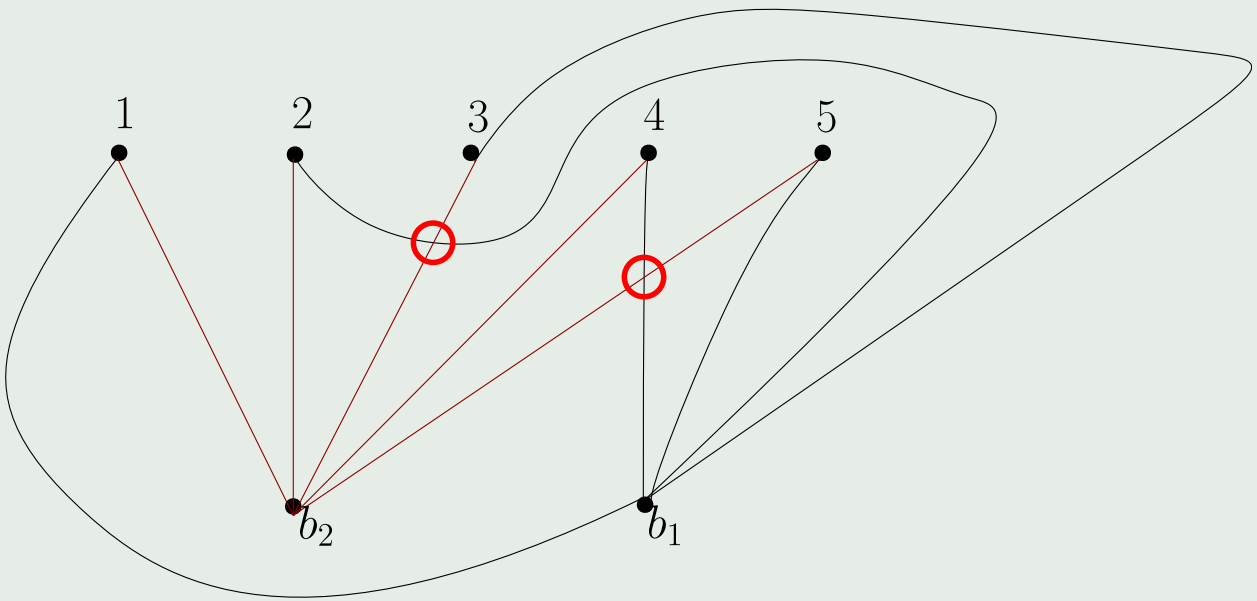

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Lower bound on crossing number

Let Z_m be the set of m-cycles, and let $Q \in \mathbb{R}^{Z_m \times Z_m}$ with

$$Q_{\sigma,\tau} := \text{min. \#swaps of adjacent elements of } \sigma \text{ to change } \sigma \text{ into } \tau^{-1}.$$

Lower bound: $\mathrm{cr}(K_{m,n}) \geq \sum_{i<j} Q_{\gamma(b_i),\gamma(b_j)}$.

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Theorem (de Klerk, Maharry, Pasechnik, Richter, Salazar; 2006)

Define $q_m := \min\left\{x^{\mathsf{T}} Q x \mid x \in \mathbb{R}^{Z_m}_{\geq 0},\ \mathbf{1}^{\mathsf{T}} x = 1\right\}$. Then

$$\mathrm{cr}(K_{m,n}) \geq \tfrac{1}{2} n^2 q_m - \tfrac{1}{2} n \lfloor \tfrac{1}{4}(m-1)^2 \rfloor \text{ for all } m, n.$$

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Semidefinite relaxation of q_m

$$q_m \geq \alpha_m := \min\{\langle X, Q\rangle \mid X \in \mathbb{R}^{Z_m \times Z_m}_{\geq 0},\ \langle X, J\rangle = 1,\ X \text{ positive semidefinite}\}.$$

Exploiting symmetry for computing α_m

Let $G_m := S_m \times \{-1, +1\}$. Then G_m acts on Z_m via

$$(\pi, \varepsilon) \cdot \sigma = \pi \sigma^{\varepsilon} \pi^{-1}, \qquad \text{for } (\pi, \varepsilon) \in G_m \text{ and } \sigma \in Z_m.$$

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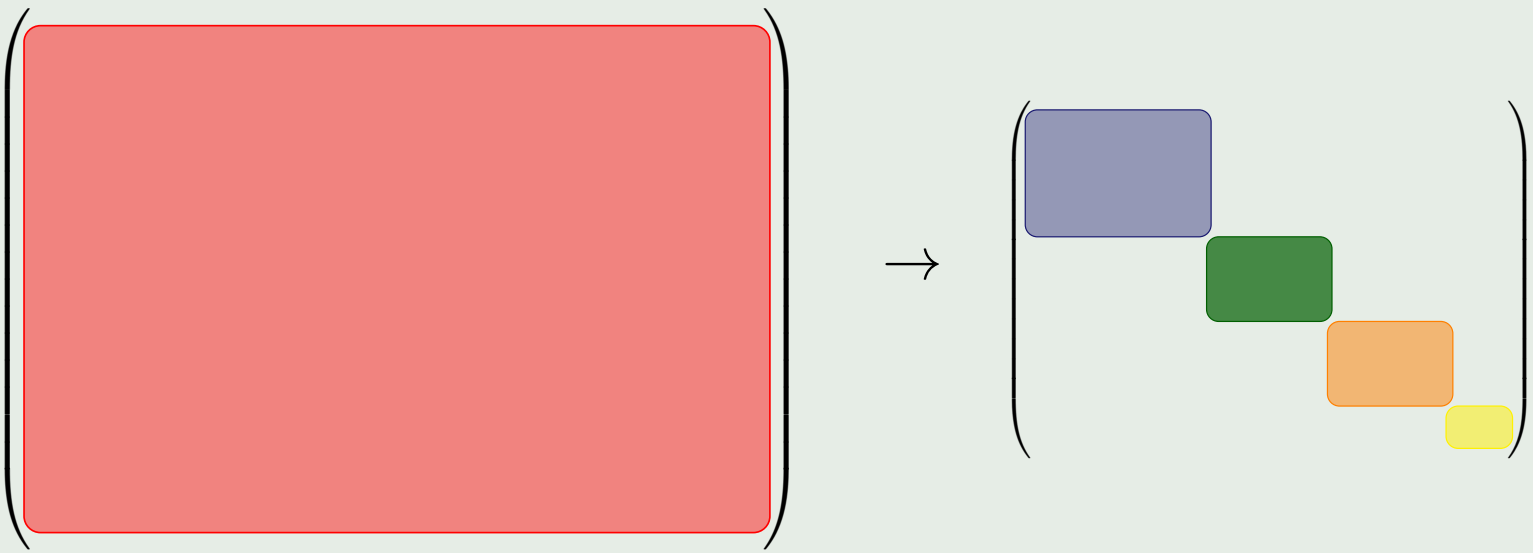
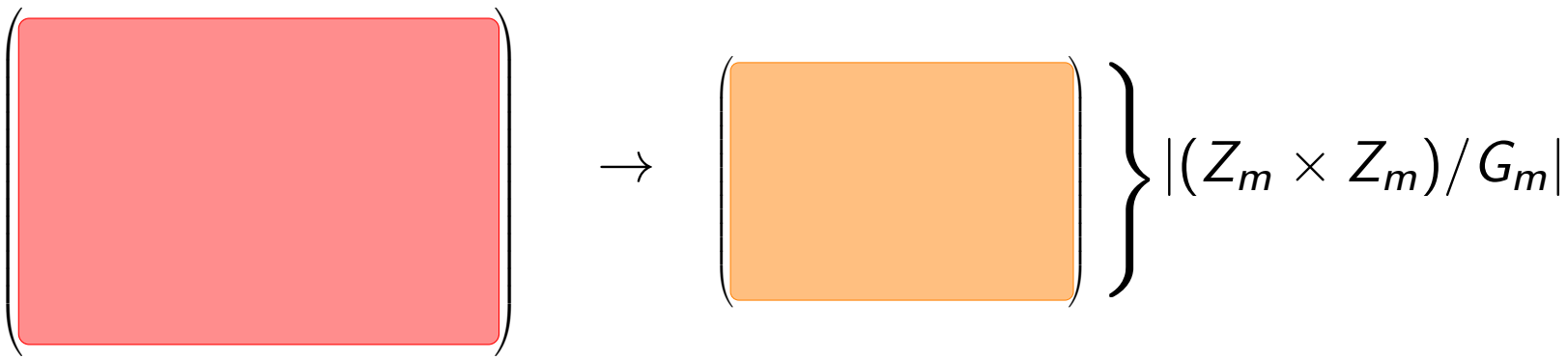
Crucial fact

Restriction of search space: find X in algebra of G_m-invariant matrices.
There exists a *block-diagonalization* of this algebra. (Artin-Wedderburn)



Results

- De Klerk, Maharry, Pasechnik, Richter, Salazar (2006) compute α_7.
- Dobre and Vera (2015) compute a better bound on q_7.
- De Klerk, Pasechnik, Schrijver (2007) compute α_8 and α_9, with a symmetry reduction using the regular representation.



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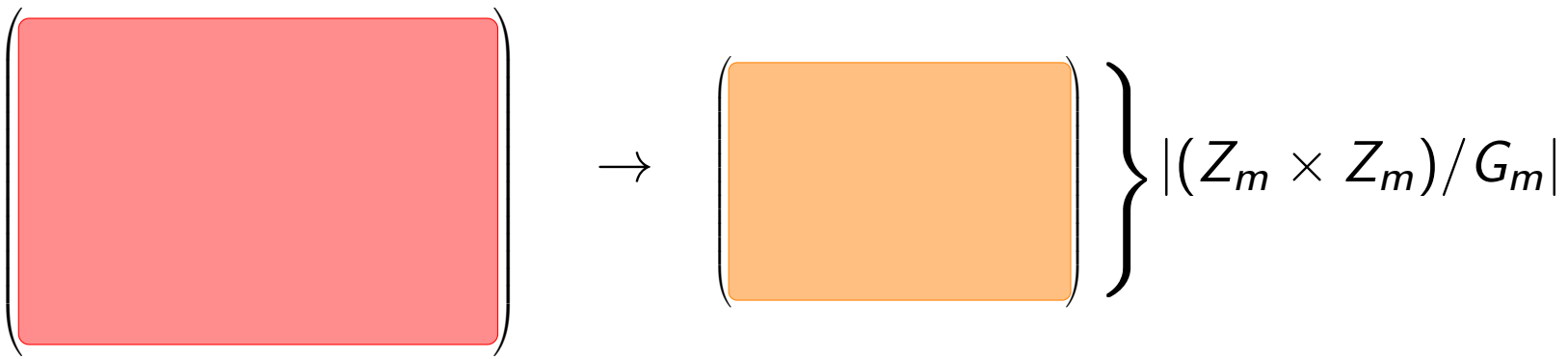
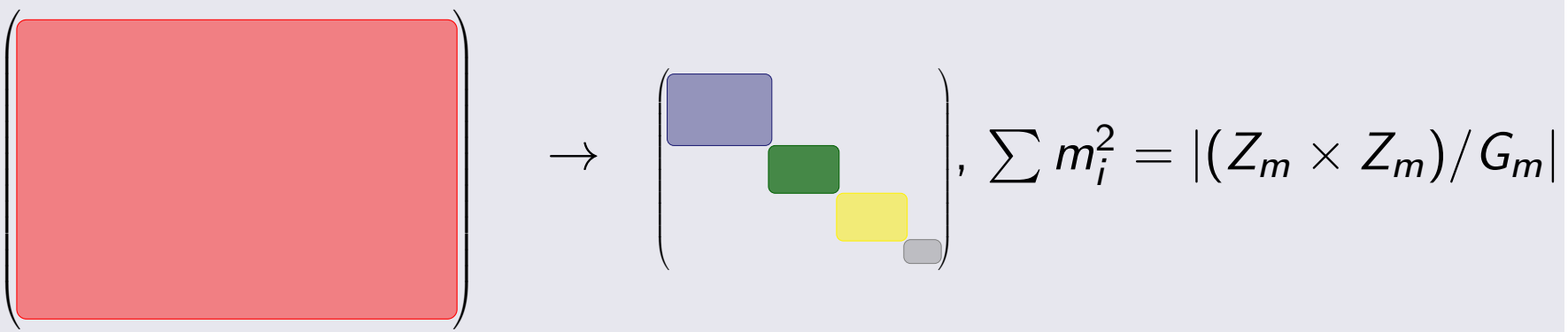


- With Daniel Brosch, we compute α_{10} using the full symmetry.



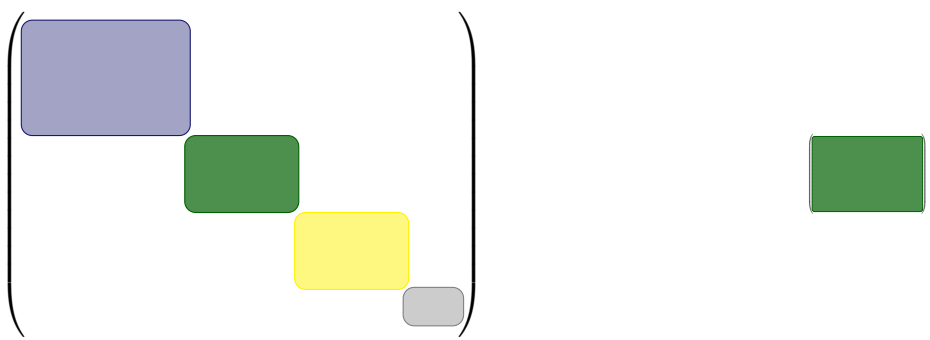
- We also give a lower bound β_m on α_m and compute it for $m \leq 13$.

The relaxation β_m

Observation

Requiring only one specific block of size $\lfloor \frac{m-1}{2} \rfloor$ to be PSD gives a good lower bound on α_m.

- Dual of β_m: few variables, small matrix block, many linear constraints.
- We solve the program without constraints, and iteratively add the most violated constraint until all constraints are satisfied.



Computational results

Theorem (Brosch, P.; 2022+)

$$\mathrm{cr}(K_{10,n}) \geq 4.8345n^2 - 10n,$$
$$\mathrm{cr}(K_{11,n}) \geq 5.9088n^2 - 12.2222n,$$
$$\mathrm{cr}(K_{12,n}) \geq 7.0906n^2 - 14.6666n,$$
$$\mathrm{cr}(K_{13,n}) \geq 8.3798n^2 - 17.3333n.$$

n	best lower bound previously known	new lower bound	$Z(n,n)$
10	384	**388**	400
11	581	**589**	625
12	846	**865**	900
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Table: New bounds on $\mathrm{cr}(K_{n,n})$

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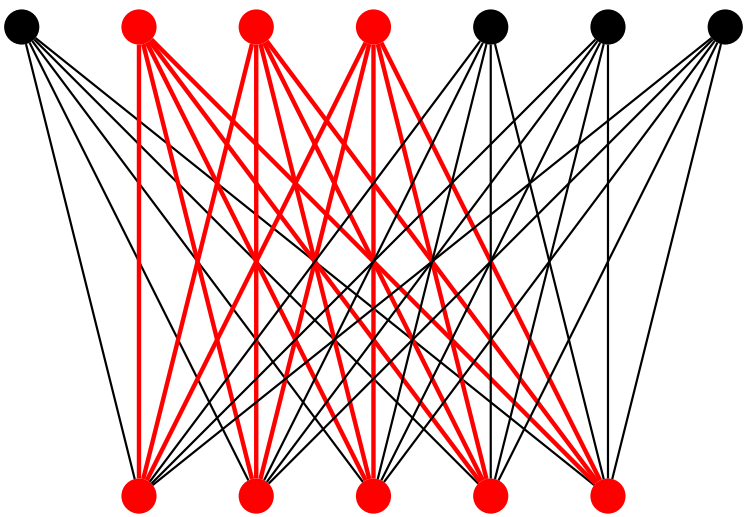

$$\mathrm{cr}(K_{m,n}) \geq \frac{\mathrm{cr}(K_{k,n})\binom{m}{k}}{\binom{m-2}{k-2}}$$

$\rightsquigarrow$ lower bounds on $\mathrm{cr}(K_{m,n})$ for $m \geq 13$.

Asymptotic results

Lemma (de Klerk, Maharry, Pasechnik, Richter, Salazar; 2006)

$$\lim_{n\to\infty} \frac{\mathrm{cr}(K_{n,n})}{Z(n,n)} \geq \frac{8\alpha_k}{k(k-1)}.$$

k	$\frac{8\alpha_k}{k(k-1)}$	$\frac{8\beta_k}{k(k-1)}$
7	0.8303	0.8210
8	0.8371	0.8326
9	0.8595	0.8503
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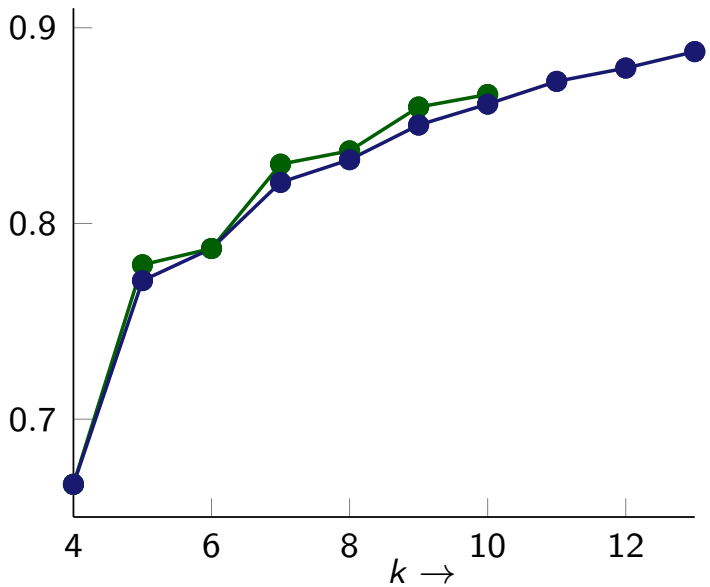

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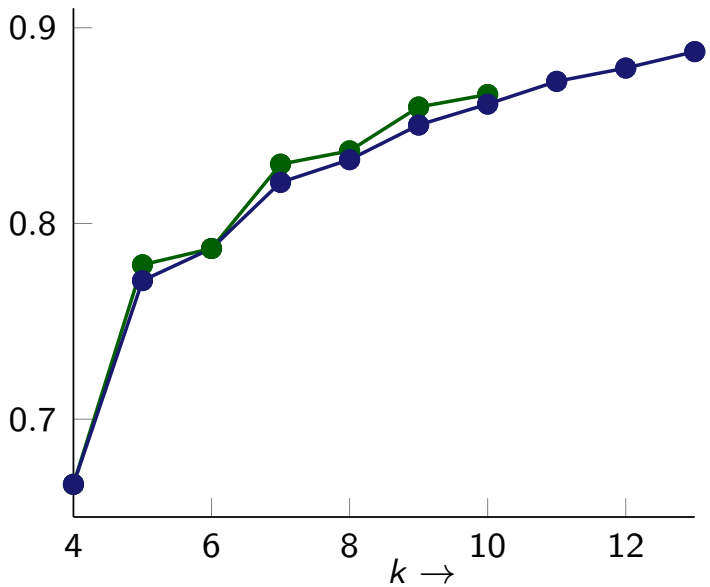

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$$\lim_{n\to\infty} \frac{\mathrm{cr}(K_{m,n})}{Z(m,n)} \geq 0.8878\frac{m}{m-1} \text{ for each } m \geq 13.$$

Summary

- Semidefinite programming gives good lower bounds on $\mathrm{cr}(K_{m,n})$.

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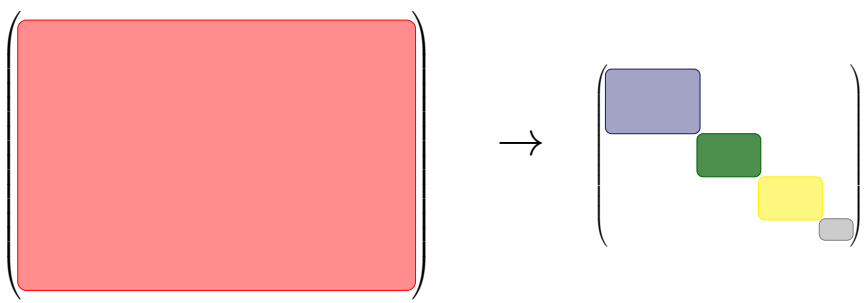

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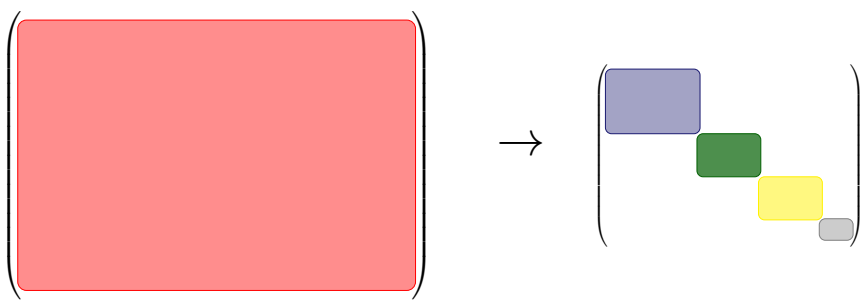

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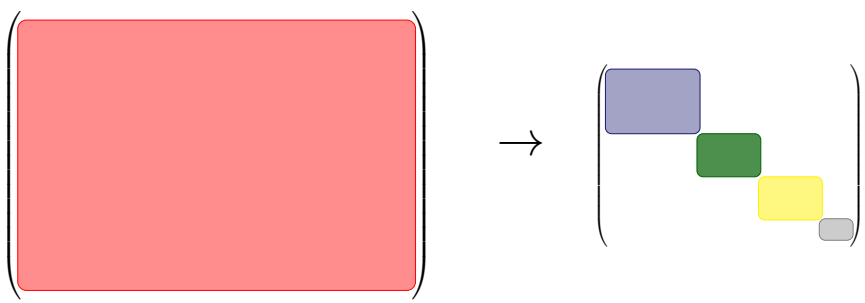

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- Possible to give optimum solution to β_m analytically?

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13	1192	**1229**	1296
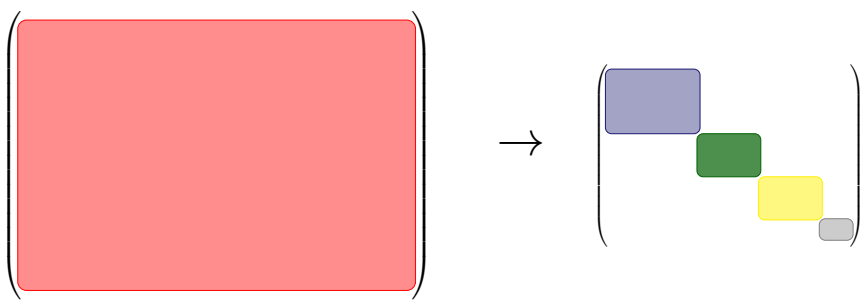
Table: New lower bounds on $\mathrm{cr}(K_{n,n})$