Cycles in Mallows random permutations

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The Mallows model

- $$\text{inv}(\pi) := |\{(i,j) : i < j \text{ and } \pi(i) > \pi(j)\}|,$$

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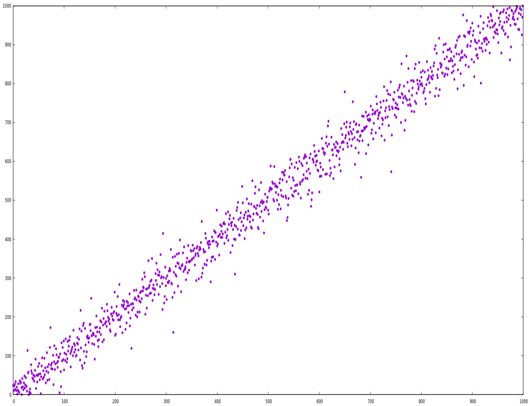
- Setting $q = 1$ we retrieve the uniform distribution on S_n
- Intuition:
  - when $0 < q < 1$ we stay "close to the identity $i \mapsto i$",
  - when $q > 1$ we stay "close to reverse map $i \mapsto n + 1 - i$".

A simulation



$(n = 1000, q = .95.)$

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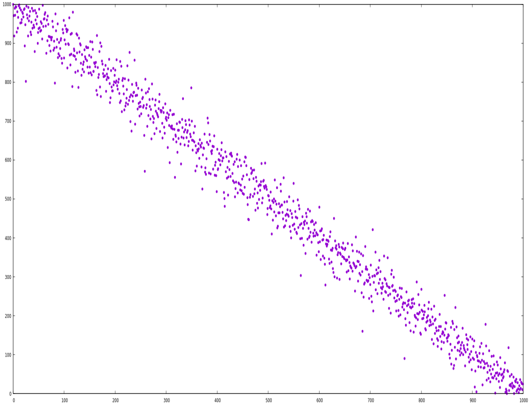

$(n = 1000, q = 1.05.)$

Background

Introduced by C.L. Mallows in 1957 in the context of "statistical ranking theory".

Since then studied in connection with mixing times of Markov chains, finitely dependent colorings of the integers, stable matchings, random binary search trees, learning theory, q-analogs of exchangeability, determinantal point processes, statistical physics, genomics.

Cycle structure

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- If Π_n selected uniformly from S_n then $C_1(\Pi_n) \to \text{Pois}(1)$ as $n \to \infty$
- In general if Π_n selected uniformly

$$(C_1(\Pi_n), \ldots, C_\ell(\Pi_n)) \to (\text{Pois}(1), \text{Pois}(1/2), \ldots, \text{Pois}(1/\ell))$$

(Gontcharoff [1941] and Kolchin [1976])

We are interested in the cycle structure

$$(C_1(\Pi_n), C_2(\Pi_n), \ldots, C_\ell(\Pi_n))$$

where $\Pi_n \sim \text{Mallows}(n, q)$ and $q \neq 1$

Cycle structure when $q < 1$

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Theorem (He 2021+, Müller+V 2022+)

Fix $0 < q < 1$ and let $\Pi_n \sim \mathrm{Mallows}(n, q)$. There exist positive constants $m_1, m_2, \dots$ and an infinite matrix $P \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ such that for all $\ell \geq 1$ we have

$$\frac{1}{\sqrt{n}}\left(C_1(\Pi_n) - m_1 n, \dots, C_\ell(\Pi_n) - m_\ell n\right) \quad \xrightarrow[n\to\infty]{d} \quad \mathcal{N}_\ell(\underline{0}, P_\ell),$$

where $\mathcal{N}_\ell(\cdot,\cdot)$ denotes the ℓ–dimensional multivariate normal distribution and P_ℓ is the submatrix of P on the indices $[\ell] \times [\ell]$.

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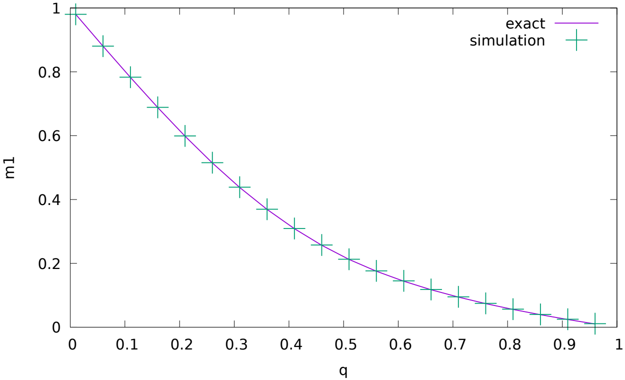
- Different behaviour from $q = 1$: Expect linear number of i–cycles as $n \to \infty$

Cycle structure when $q < 1$

m_1 is the fraction of fixed points



$m_1 \to 1$ as $q \downarrow 0$ (tends to identity perm.)
$m_1 \to 0$ as $q \uparrow 1$

$q > 1$ (cycles of even length)

Theorem (He 2021+, Müller+V 2022+)

For $q > 1$ there exist positive constants $\mu_2, \mu_4, \ldots$ and an infinite matrix $Q \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ such that for all $\ell \geq 1$ we have

$$\frac{1}{\sqrt{n}}(C_2(\Pi_n) - \mu_2 n, \ldots, C_{2\ell}(\Pi_n) - \mu_{2\ell} n) \xrightarrow{d} \mathcal{N}_\ell(\underline{0}, Q_\ell),$$

where $\mathcal{N}_\ell(\cdot,\cdot)$ denotes the ℓ–dimensional multivariate normal distribution and Q_ℓ is the submatrix of Q on the indices $[\ell] \times [\ell]$.

$q > 1$ (cycles of odd length)

Theorem (He 2021+, Müller+V 2022+)

For $q > 1$, *let* $\Pi_n \sim \mathrm{Mallows}(n, q)$

$$(C_1(\Pi_{2n+1}), C_3(\Pi_{2n+1}), \ldots) \quad \textit{converges to some distribution,}$$

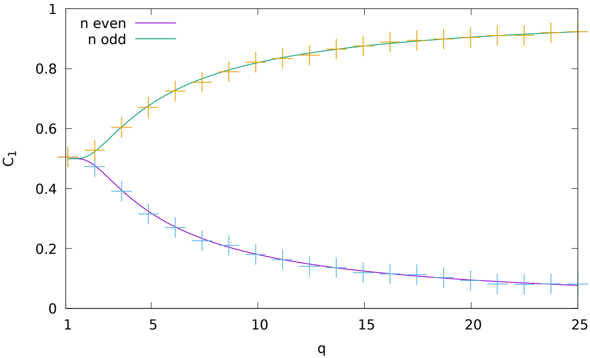
and

$$(C_1(\Pi_{2n}), C_3(\Pi_{2n}), \ldots) \quad \textit{converges to some distribution.}$$

Moreover, the two limiting distributions above are distinct for all $q > 1$.

- We give explicit limiting distributions in terms of an extension of $\mathrm{Mallows}(n, q)$ to $\mathbb{Z}$ due to Gnedin and Olshanski

Fixed points for $q > 1$



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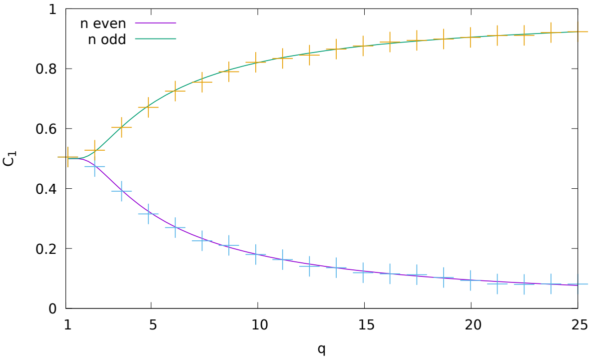

- $\mathbb{E}C_1 \to \frac{1}{2}$ as $q \downarrow 1$
- $\mathbb{E}C_1(\Pi_{2n+1}) \to 1$ as $q \to \infty$
- $\mathbb{E}C_1(\Pi_{2n}) \to 0$ as $q \to \infty$

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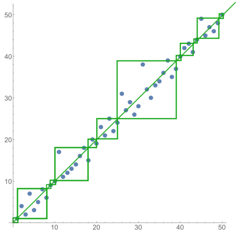

(Plot taken from [Basu+Bhatnagar 2017].)

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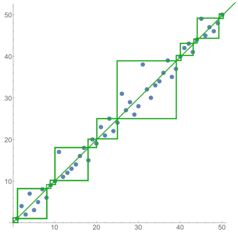

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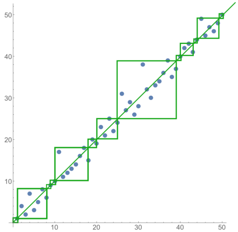

(Plot taken from [Basu+Bhatnagar 2017].)

- For large n there are many T_i such that $\Pi_n[[T_i]] = [T_i]$
- The 'renewal blocks' are approximately i.i.d.

For $q < 1$ then

$$\frac{1}{\sqrt{n}} \left(C_1(\Pi_n) - m_1 n, \ldots, C_\ell(\Pi_n) - m_\ell n \right) \quad \xrightarrow[n \to \infty]{d} \quad \mathcal{N}_\ell(\underline{0}, P_\ell),$$

follows from results on renewal processes by Gut+Janson 1983

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$$\mathrm{inv}(r_n \circ \pi) = \binom{n}{2} - \mathrm{inv}(\pi)$$

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That is

$$r_n \circ \Pi_n \overset{d}{=} \mathrm{Mallows}(n, 1/q) \quad \text{if} \quad \Pi_n \sim \mathrm{Mallows}(n, q)$$

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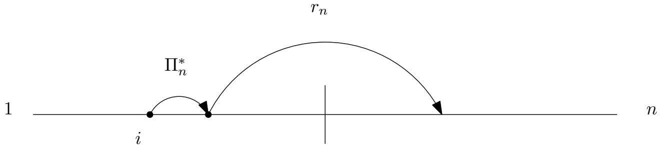

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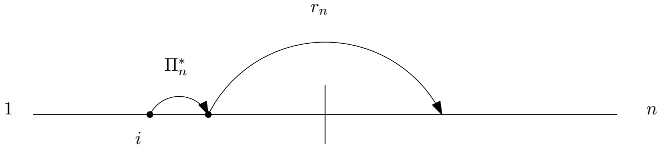

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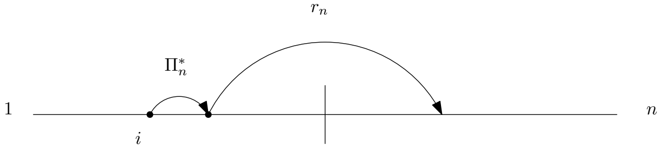

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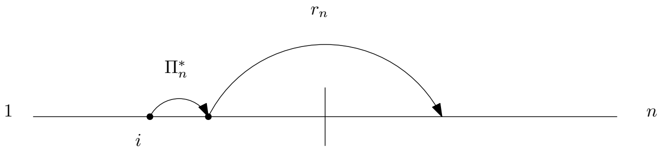

- Π_n^* 'close to identity' $\implies$ $\Pi_n^*(i)$ close to i
- $r_n \circ \Pi_n^*(i)$ far from i if i far from middle of $[n]$
- Fixed points for $q > 1$ must occur close to the mid point of $[n]$

$q > 1$ even cycles

We do not see the same obstruction for even cycles that we saw for odd cycles:

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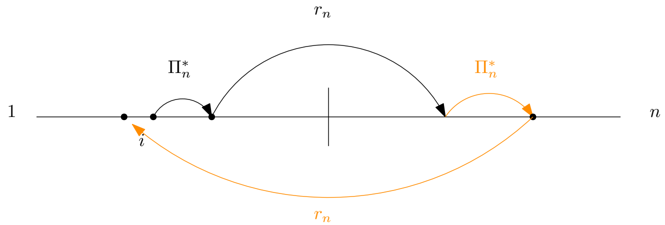
Take again $\Pi_n^* \sim \text{Mallows}(n, 1/q)$ and $\Pi_n := r_n \circ \Pi_n^*$



$q \to \infty$

- $\mathbb{E}C_1(\Pi_{2n+1}) \to 1 \qquad$ as $q \to \infty$
- $\mathbb{E}C_1(\Pi_{2n}) \to 0 \qquad$ as $q \to \infty$

Is this reasonable?

$q \to \infty$

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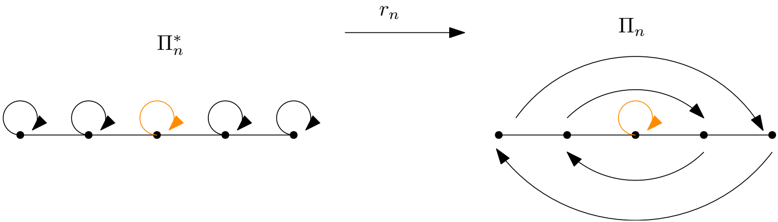
Is this reasonable?

- As $q \to \infty$ we have $1/q \to 0$ so that $\Pi_n^* \sim \text{Mallows}(n, 1/q)$ 'gets very close to the identity'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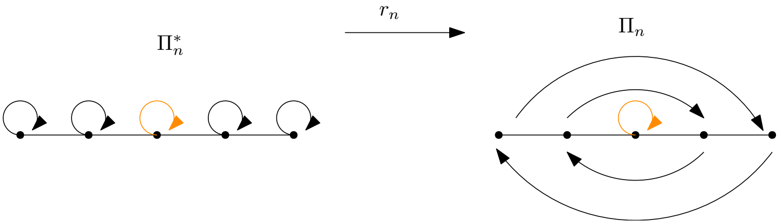
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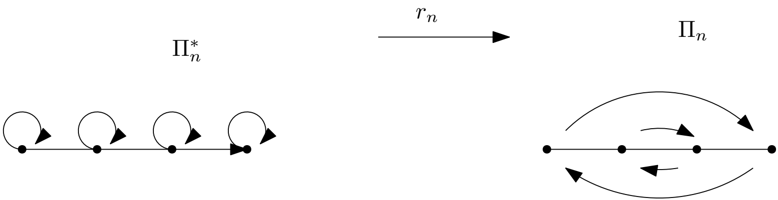
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Thanks for your attention. Questions?