On a nonabelian Kneser theorem

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Kneser's theorem in abelian groups

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Generalizing the Cauchy–Davenport theorem in $\mathbb{Z}/p\mathbb{Z}$, Kneser in 1953 proved the following.

Theorem (Kneser's addition theorem - 1953)

Let A, B be finite subsets of an abelian group G, and let H be the stabilizer of $A + B$, i.e. the largest subgroup $H < G$ such that $A + B + H = A + B$. Then

$$|A + B| = |A + H| + |B + H| - |H|.$$

In particular, if $|A + B| < |A| + |B| - 1$, then H is nontrivial.

The situation in general groups: counterexamples

Great interest to find a suitable generalization of this to general groups, where we usually talk about the **product set** AB instead.

Natural question: Can we generalize Kneser directly, replacing periodicity by left- or right-periodicity?

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Natural question: Can we generalize Kneser directly, replacing periodicity by left- or right-periodicity?

No! Olson in 1986 constructed examples showing that a direct generalization of Kneser's theorem does not hold.

Theorem (Olson 1986)

There exists a finite group G and subsets $A, B \subset G$ such that

$$|AB| < |A| + |B| - 1,$$

but for any nontrivial subgroup $H < G$ it holds that

$$AB \notin \{HAB, AHB, ABH\}.$$

The situation in general groups: positive results

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Theorem (Hamidoune 2013)

Let $A \subset G$ be nonempty and finite, and let $G' = \langle A^{-1}A \rangle$. If

$$\max\{|A^{-1}A|, |AA^{-1}|\} \leq \min\{2|A| - 2, |G'| - 1\},$$

then there exists a nontrivial subgroup $H < G'$ and an element $a \in A$ such that $A^{-1}HA = A^{-1}A \cup a^{-1}Ha$ or $AHA^{-1} = AA^{-1} \cup aHa^{-1}$.

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Theorem (DeVos, preprint)

Let $A, B \subset G$ finite and nonempty. If $|AB| < |A| + |B| - 1$, then there exists a nontrivial subgroup $H < G$ such that the following holds. For every $x \in AB$ there exists a $y \in G$ such that

$$xy^{-1}Hy \subset AB.$$

In particular, AB is the union of left-cosets of (possibly distinct) conjugates of H.

Recasting to graphs and the isoperimetric method

Let $\Gamma = (V, E)$ be a vertex-transitive digraph with a loop at every vertex (*reflexive*). Some definitions:

- **Boundary of X ⊂ V:** $\partial_\Gamma(X) = N_\Gamma(X) \setminus X$.
- **Connectivity:** $\kappa(\Gamma) = \min\{|\partial_\Gamma(X)| : X \subset V \text{ finite and } N_\Gamma(X) \neq V\}$.
- **Fragment:** Finite $F \subset V$ s.t. $N_\Gamma(F) \neq V$ and $|\partial_\Gamma(F)| = \kappa(\Gamma)$.
- **Atom:** Fragment of minimal cardinality $\alpha(\Gamma)$.

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The isoperimetric method (name coined by Hamidoune): Obtain precise structural information on extremal objects like atoms and fragments, and then leverage this to prove statements on *every* set with a small but not minimal boundary.

The connection to groups

Suppose G is a group and $1 \in S \subset G$ a finite generating set. The Cayley digraph $Cay(G, S)$ has vertex set G and edges (g, gs) for every $g \in G$ and $s \in S$.

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- For $A \subset G$, the product set AS is exactly the neighborhood $N(A)$ in $Cay(G, S)$.
- Fact: The atoms of $Cay(G, S)$ or $Cay(G, S^{-1})$ will be left-cosets of a subgroup $H < G$.

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From now, for a digraph Γ, we always denote by Γ^{-1} the digraph obtained by reversing all the edges of Γ.

Our result

Almost periodicity: If $\mathbf{Z} \subset 2^{V(\Gamma)}$ is a system of imprimitivity of $\mathrm{Aut}(\Gamma)$, we say that a set $A \subset V$ is almost periodic with respect to $\mathbf{Z}$ if $A \cap B \in \{\emptyset, B\}$ for all but at most one $B \in \mathbf{Z}$.

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Theorem (Serra, W. 2022+)

Let Γ be a finite vertex-transitive reflexive digraph with degree $d(\Gamma)$. If

$$\kappa(\Gamma) < d(\Gamma) - 1 \quad \text{and} \quad \alpha(\Gamma) = \alpha(\Gamma^{-1}),$$

then for every vertex $x \in V(\Gamma)$, the alternated second neighborhood $N_\Gamma(N_{\Gamma^{-1}}(x))$ is almost periodic with respect to the atoms of Γ^{-1}.

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Corollary

Let A be a nonempty subset of a finite group G such that $a^{-1}A = A^{-1}a$ for some $a \in A$. If

$$A^{-1}A < 2|A| - 1,$$

then there exists a nontrivial subgroup $H < G$ and an element $x \in A^{-1}A$ such that $A^{-1}AH = A^{-1}A \cup xH$.

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Theorem (No periodicity)

There exists a finite group G and a generating set $1 \in S \subset G$ such that $\Gamma = Cay(G, S)$ satisfies $\alpha(\Gamma) = \alpha(\Gamma^{-1})$ and $\kappa(\Gamma) < |S| - 1$ but

$$N_\Gamma(N_{\Gamma^{-1}}(1)) = S^{-1}S$$

is not periodic with respect to the atoms of $\Gamma^{-1} = Cay(G, S^{-1})$, which are left-cosets of a nontrivial subgroup $H < G$. In fact,

$$S^{-1}S \notin \{HS^{-1}S, S^{-1}HS, S^{-1}SH\}.$$

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For every $n \in \mathbb{Z}^+$ there exists an infinite vertex-transitive reflexive undirected graph Γ such that for every $x \in V(\Gamma)$,

$$N(N(x)) \cap A \notin \{\emptyset, A\}$$

for $2n - 1$ atoms A of Γ.

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Concluding comments

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- Can the *non-periodicity* construction be improved such that

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- Can the infinite counterexample be constructed as a Cayley graph? If so, same question as before: Can we get the result for any nontrivial subgroup instead of just atoms?
- In the other direction, maybe the opposite is true? For instance, in the non-periodicity construction, $S^{-1}S$ is (left, middle, and right)-periodic with respect to the intersection of the atoms of Γ and Γ^{-1}. Does something like this always work?

Thank you all for your attention!