Anti-Ramsey numbers for vertex-disjoint triangles

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Joint work with Shenggui Zhang[a], Binlong Li[a] and Jimeng Xiao[a]

May 12, 2022

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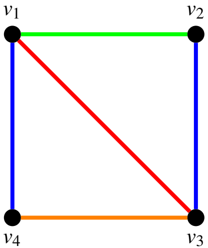

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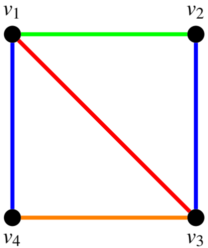

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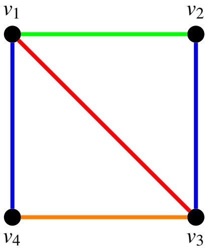

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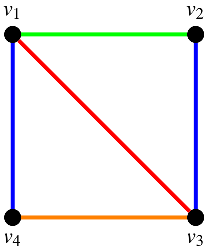

$C_3 = v_1v_2v_3v_1$: rainbow

$C_4 = v_1v_2v_3v_4v_1$: not rainbow

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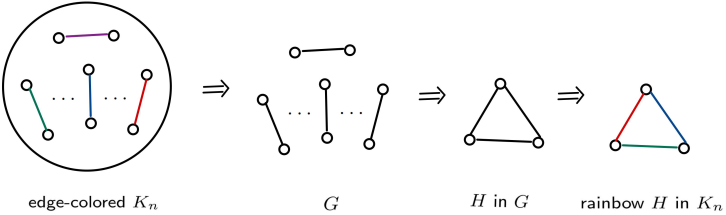
- **Anti-Ramsey number** $ar(n,H)$: the maximum number of colors in an edge-coloring of K_n with no rainbow copy of H
- By taking one edge of each color in an edge-coloring of K_n, one can show that $ar(n,H) \leq ex(n,H)$.



Anti-Ramsey numbers

- **Anti-Ramsey numbers were introduced by Erdős, Simonovits and Sós. They showed that these are closely related to Turán numbers. Since then numerous results were established for a variety of graphs, including, among others, cycles, cliques, paths, matchings and trees.**

P. Erdős, M. Simonovits and V.T. Sós, Anti-Ramsey theorems, in Infinite and Finite sets (Colloq. Keszthely 1973), Colloq. Math. Soc. János Bolyai 10 (1975) 633–643.

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- **An interesting open problem concerning anti-Ramsey numbers is the determination of the anti-Ramsey number of cycles.**

Anti-Ramsey numbers for cycles

Conjecture (Erdős et al., 1973)

For all $n \geq l \geq 3$, $ar(n, C_l) = (\frac{l-2}{2} + \frac{1}{l-1})n + O(1)$.

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- $l = 3$ (Erdős, Simonovits and Sós, 1973)
- $l = 4$ (Alon, 1983)
- $l \leq 7$ (Jiang, Schiermeyer and West, 2004)

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- **In 2005, Montellano-Ballesteros and Neumann-Lara proved the above conjecture.**

J.J. Montellano-Ballesteros and V. Neumann-Lara, An anti-Ramsey theorem on cycles, Graphs Combin., 21 (2005) 343–354.

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For any integer n and k with $n \geq 3k$ and $k \geq 2$,

$$\binom{2k-2}{2} + (2k-2)(n-2k+2) + 1 \leq ar(n, \Omega_k) \leq g(n,k) - 1.$$

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- **We improve the result of Yuan and Zhang from n sufficiently large to $n \geq 2k^2 - k + 2$, and give some bounds and exact values for the other n.**

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- **If $n < 3k$, then K_n does not have enough vertices for k vertex-disjoint triangles, which implies that a rainbow K_n does not have kC_3. Hence $ar(n, kC_3) = \binom{n}{2}$ when $n < 3k$.**

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- **For $n \geq 3k$, the question becomes nontrivial. We first give a lower bound for $ar(n, kC_3)$.**

Anti-Ramsey numbers for vertex-disjoint triangles

Theorem (Wu, Zhang, Li and Xiao, 2022+)

$ar(n, kC_3) \geq \max\{\binom{3k-1}{2} + n - 3k + 1, \left\lfloor \frac{(n-k+2)^2}{4} \right\rfloor + (k-2)(n-k+2) + \binom{k-2}{2} + 1\}$ *for all* $n \geq 3k$.

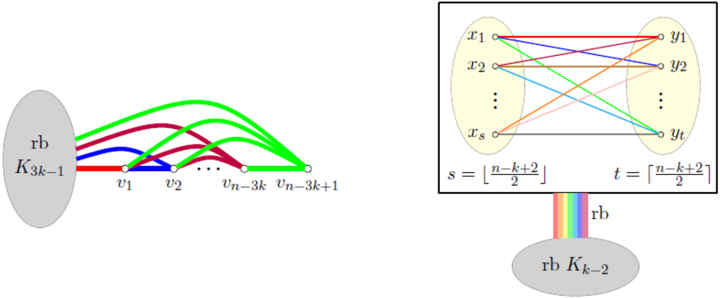

Fig. Two colorings of K_n.

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Conjecture (Wu, Zhang, Li and Xiao, 2022+)

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Vertex-disjoint triangles in general graphs

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- **They enrich the conclusions of the existence of vertex-disjoint cycles in general graphs.**

Vertex-disjoint triangles in general graphs

Lemma (Wu, Zhang, Li and Xiao, 2022+)

Let G be a graph on n vertices with $n \geq 3k$. If $\delta(G) \geq \frac{n+k-1}{2}$ and G contains no kC_3's, then there exists a partition $V(G) = V_1 \cup V_2 \cup V_3$, so that $|V_1| = k-1$, $|V_2| = |V_3| = \frac{n-k+1}{2}$, V_2 and V_3 are independent sets, and any two vertices in different parts are adjacent, unless G is isomorphic to G' when $n = 10$ and $k = 3$.

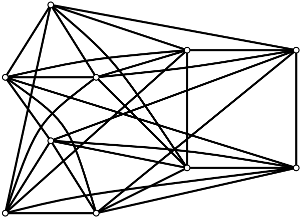

Fig. An exceptional graph G'.

Vertex-disjoint triangles in general graphs

Lemma (Wu, Zhang, Li and Xiao, 2022+)

Let G be a graph on n vertices with $n \geq 3k+1$. If $\delta(G) \geq \frac{n+k-2}{2}$ and G contains a $(k-1)C_3$ but not kC_3's, then there exists a subgraph $H \subseteq G$ such that $|V(H)| = 3k-3$, H contains a $(k-1)C_3$ and $G - H$ is a complete bipartite graph whose two parts are at least 2 in size.

Thank you for your attention!